

ARTÍCULO

Risk optimal single-object auctions

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Abstract We analyze the preferences of a risk-averse auctioneer over several auction mechanisms with risk-neutral and symmetric bidders. We obtain the value at risk (VaR) for auctioneer revenue in auction mechanisms belonging to a parametric family which includes two classic mechanisms, the first-price auction and second-price auction. By calculating the VaR for revenue an auctioneer can estimate the amount that will be lost within a given confidence level, depending on the number of bidders and the auction mechanism chosen. The contribution of this paper is the calculation of the VaR for auctioneer revenue in some common auction mechanisms that yield the same expected revenue, including first-price auction and second-price auction and the following mechanisms: Santa Claus auction, sad-loser auction and all-pay auction. We describe how to quantify the maximum loss for an auctioneer at a given probability. We study the value at risk of the auctioneer as a criterion to determine which auctions would best suit the auctioneer's interests.

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Modelos de subasta de un objeto con riesgo óptimo

Resumen Analizamos las preferencias de un subastador con aversión al riesgo en diferentes modelos de subasta con participantes neutrales al riesgo y simétricos. Obtenemos el valor en riesgo (VaR) de los ingresos del subastador en los mecanismos de subasta que pertenecen a una familia paramétrica que incluye dos mecanismos clásicos: la subasta de primer precio y la subasta de segundo precio. Mediante el cálculo del VaR de los ingresos, un subastador puede estimar la cantidad que se perderá dentro de un nivel de confianza dado, dependiendo del número de participantes y del mecanismo de subasta elegido. La novedad de este artículo es el cálculo del VaR de los ingresos del subastador en algunos mecanismos de subasta comunes, que producen los mismos ingresos esperados y que incluyen la subasta de primer precio y la subasta

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de segundo precio, así como los siguientes mecanismos: subasta «Santa Claus», subasta «Perdedores Tristes» y subasta «Todos pagan». Describimos cómo cuantificar la pérdida máxima de un subastador con una probabilidad dada. Estudiamos el valor en riesgo del subastador como un criterio para determinar qué subastas se adaptarían mejor a los intereses del subastador.
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1. Introduction

Under some conditions, when a single object is auctioned, two classical auction mechanisms, the first-price auction (FPA) and second-price auction (SPA), generate the same expected payment by bidders and therefore the same expected revenue for the auctioneer (Vickrey, 1961). In fact, the Revenue Equivalence Theorem (Myerson, 1981) is verified if two auction mechanisms have the same allocation rule and produce the same bidder payment with valuation 0. If the auctioneer takes only the expected revenue into account (risk-neutral auctioneer) then every auction mechanism that satisfies the Revenue Equivalence Theorem is equally attractive to the auctioneer. However, every risk-averse auctioneer would prefer the auction mechanism with the lowest variance (assuming, of course, that the bidders are risk-neutral).

To establish a preference between the classical auction mechanisms, their variability (usually their variance) is often compared. Vickrey (1961) calculated the variance for FPA and English auctions. Waehrer et al. (1998) proved that a risk-averse auctioneer prefers FPA to SPA and SPA to an English auction. Beltrán and Santamaría (2006) used simulation to analyze the variation for several auction mechanisms with the same expected revenue. Krishna (2002) proved that the price distribution in SPA is a mean-preserving spread of the price distribution in FPA.

The main problem with variance is that it does not indicate the direction of revenue deviations: revenue can be volatile and can suddenly take high values. An auctioneer is not affected if revenue is higher than expected; however, much lower than expected revenues could lead to bankruptcy. Hence, we are interested in obtaining a measure of the risk of losses for each auction mechanism.

In the literature we can find a number of papers on the optimal choice or design of auctions, usually from the point of view of the expected revenue (see, for example, Myerson, 1981; Riley and Samuelson, 1981). For this reason, if the auctioneer only takes into account the expected revenue, then every auction mechanism satisfying the Revenue Equivalence Theorem is identical for her. Consequently, the auctioneer should take into account other criteria when making her decision. A possible criterion is collusion (Robinson, 1985), another is variability (Vickrey, 1961), and other criteria could also be used. We aim at giving the auctioneer a second criterion for choosing an appropriate auction mode. Therefore, the original contribution of this paper is the analysis of risk measures applied to auctions.

Value at risk (VaR) is a measure of the worst loss at a given confidence level and reflects how much can be lost with respect to expected revenue at a certain probability (Holton, 2004). We must bear in mind that although expected auctioneer revenue is a good long-term indicator, a sudden

and unexpected decrease in actual revenue could lead to bankruptcy. The VaR for auctioneer revenue at a given confidence level quantifies the maximum loss compared to expected revenue. Thus, if an auctioneer knows in advance that the possible maximum loss is greater than what can be absorbed economically, the auction mechanism can be changed or the number of bidders can be increased to reduce this risk. We therefore believe that VaR can be very useful for auctions that, despite being equally attractive in terms of expected revenue, are different in terms of loss risk.

The contribution of this paper is the calculation of the VaR for auctioneer revenue in some common auction mechanisms that yield the same expected revenue, including FPA and SPA and the following mechanisms: Santa Claus auction (SCA), sad-loser auction (SLA) and all-pay auction (APA). Thus, we quantify the maximum loss for an auctioneer at a given probability.

We assume the following hypotheses:

There are $n \geq 2$ risk-neutral bidders competing to buy a single object. The valuation of bidder $i \in \{1, \dots, n\}$ is θ_i . Each type θ_i , which is information known only to bidder i , is an independent realization of a uniformly distributed continuous random variable Θ_i in $[0, 1]$. Only bidder i observes the realization of θ_i , which reflects the uncertainty that bidder j has about the valuation of bidder i .

Each bidder $i \in \{1, \dots, n\}$ simultaneously and independently submits a bid $b_i \in [0, 1]$ specifying the maximum unit-price offer at which he is willing to buy the object.

A strategy for bidder $i \in \{1, \dots, n\}$ is a function $b_i(\theta_i): [0, 1] \rightarrow [0, 1]$.

We refer to this model as the *symmetric model*.

Once the auctioneer has received all the bids, the highest bidder wins the object. The price paid by each bidder depends on the auction mechanism adopted for the transaction. Each aspect of this model and the auction mechanism chosen is assumed to be common knowledge. This situation is modeled as a single-object auction as an n -person game with incomplete information (Harsanyi, 1967-1968) under the assumption of symmetric and risk-neutral bidders with independent and uniformly distributed $[0, 1]$ values.

The rest of the paper is organized as follows. In Section 2 we define the parametric family of auction mechanisms for selling one object, and obtain the Bayesian Nash equilibrium for each auction mechanism in the parametric family. In Section 2.1. we obtain the distribution function of the random variable that gives the auctioneer revenue at the equilibrium in each auction mechanism in the parametric family, we prove that each auction mechanism in the parametric family verifies a revenue equivalence theorem and we calculate the variance and the VaR of the auctioneer revenue. In Section 2.2. we give the VaR for auctioneer revenue for different numbers of bidders and different auction mechanisms. In Section 3 we analyze other auction

mechanisms by comparing them with regard to their VaR for auctioneer revenue. Finally, Section 4 gives the conclusions.

2. First-second-price auction family

The two classic auction mechanisms are FPA and SPA. In FPA the winner pays the winning bid b_i . In SPA the winner pays the second highest bid. Here we consider a parametric family of auction mechanisms that contains FPA and SPA as particular cases. This family, called the first-second-price parametric family (*FSP*), is a set of auction mechanisms A^γ whose profit function for bidder i is:

$$B_i^\gamma(\theta_i, b_1, \dots, b_n) = \begin{cases} \theta_i - \gamma b_i - (1-\gamma) \max_{j \neq i} b_j & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } b_i < \max_{j \neq i} b_j \end{cases}; \gamma \in [0, 1]$$

That is, if i bids the highest amount, then i buys the object and possibly pays different prices: an amount γ at price b_i and the remaining amount $(1-\gamma)$ at the second highest bid, $\max_{j \neq i} b_j$. In particular, if $\gamma = 1$ we have the FPA and if $\gamma = 0$, we have the SPA.

We can assume, given the symmetry of the model, that $b_i(\theta_i) = b(\theta_i)$ is the bid function used in the equilibrium by all bidders where $b(\theta_i)$ is a strictly monotone and differentiable function. The following proposition gives the Bayesian Nash equilibrium for any auction mechanism belonging to *FSP*.

Proposition 1. *If the hypotheses for the symmetric model are verified and an auction mechanism $A^\gamma \in FSP$ is used, then the unique symmetric Bayesian Nash equilibrium $(b^\gamma(\theta_1), \dots, b^\gamma(\theta_n))$ with the condition $\lim_{\theta_i \rightarrow 0} b^\gamma(\theta_i) = 0$, is:*

$$b^\gamma(\theta_i) = \frac{n-1}{n-1+\gamma} \theta_i$$

Proof. Let $\Upsilon = \text{Max}\{\Theta_1, \dots, \Theta_{i-1}, \Theta_{i+1}, \dots, \Theta_n\}$ be the highest-order statistic for the other $n-1$ bidder types. Then the profit for bidder i is:

$$B_i^\gamma(\theta_i, b(\theta_1), \dots, b(\theta_n)) = \begin{cases} \theta_i - \gamma b_i - (1-\gamma)b(\Upsilon) & \text{if } b^{-1}(b_i) > \Upsilon \\ 0 & \text{if } b^{-1}(b_i) < \Upsilon \end{cases}; \gamma \in [0, 1]$$

Bidder i knows his own type θ_i , but Υ is a random variable, so the expected profit for bidder i is given by:

$$\begin{aligned} BM(\theta_i, b_i, b) &= \int_0^1 B_i^\gamma(\theta_i, b(\theta_1), \dots, b(\theta_n)) f(\Upsilon) d\Upsilon = \\ &= \int_0^{b^{-1}(b_i)} (\theta_i - \gamma b_i - (1-\gamma)b(\Upsilon)) (n-1) \Upsilon^{n-2} d\Upsilon = \\ &= (n-1) \left((\theta_i - \gamma b_i) \frac{(b^{-1}(b_i))^{n-1}}{n-1} - (1-\gamma) \int_0^{b^{-1}(b_i)} b(\Upsilon) \Upsilon^{n-2} d\Upsilon \right) \end{aligned}$$

Thus, b_i is the best bid for bidder i if it maximizes the expected profit, given type θ_i . The derivative with respect to b_i is:

$$\begin{aligned} \frac{\partial}{\partial b_i} BM(\theta_i, b_i, b) &= \\ &= (n-1) \left(-\gamma \frac{(b^{-1}(b_i))^{n-1}}{n-1} + (\theta_i - \gamma b_i) (b^{-1}(b_i))^{n-2} \frac{d}{db_i} (b^{-1}(b_i)) - \right. \\ &\quad \left. - (1-\gamma) b_i (b^{-1}(b_i))^{n-2} \frac{d}{db_i} (b^{-1}(b_i)) \right) = \\ &= -\gamma (b^{-1}(b_i))^{n-1} + (n-1) (\theta_i - b_i) (b^{-1}(b_i))^{n-2} \frac{d}{db_i} (b^{-1}(b_i)) \end{aligned}$$

As $b^{-1}(b_i) = \theta_i \Leftrightarrow b_i = b(\theta_i)$, replacing and setting the above equation equal to zero, we obtain the following differential equation:

$$\Upsilon \theta_i \frac{d}{d\theta_i} b^\gamma(\theta_i) + (n-1) b^\gamma(\theta_i) = (n-1) \theta_i$$

Then the solution satisfying the condition $\lim_{\theta_i \rightarrow 0} b^\gamma(\theta_i) = 0$ is:

$$b^\gamma(\theta_i) = \frac{n-1}{n-1+\gamma} \theta_i.$$

2.1. Statistical properties of the revenue

Once the Bayesian Nash equilibrium has been determined, we can calculate the distribution, expectation, variance and VaR of the auctioneer revenue. It is easy to find that the expected revenue is independent of γ , which means that there is a revenue equivalence result for every auction mechanism belonging to *FSP*.

Let X^γ be the random variable that gives the auctioneer revenue at equilibrium in an auction mechanism $A^\gamma \in FSP$:

$$X^\gamma = \gamma b^\gamma(\Upsilon) + (1-\gamma) b^\gamma(Z) = \gamma \frac{n-1}{n-1+\gamma} \Upsilon + (1-\gamma) \frac{n-1}{n-1+\gamma} Z,$$

where Υ is the highest-order and Z is the second-highest-order statistic of $\{\Theta_1, \dots, \Theta_n\}$. The variable X^γ is distributed in $\left[0, \frac{n-1}{n-1+\gamma}\right]$ and its probability distribution function is given by:

$$\begin{aligned} F^\gamma(x) &= \left(\frac{(n-1+\gamma)^n}{\gamma(n-1)^n} x^n \right) I_{\left[0, \frac{\gamma(n-1)^n}{n-1+\gamma}\right]}(x) + \\ &+ \frac{1}{\gamma(n-1)^n} \left(((n-1+\gamma)x)^n - \frac{((n-1+\gamma)x - (n-1)\gamma)^n}{(1-\gamma)^{n-1}} \right) I_{\left[\frac{\gamma(n-1)^n}{n-1+\gamma}, \frac{n-1}{n-1+\gamma}\right]}(x) \\ &+ I_{\left[\frac{n-1}{n-1+\gamma}, \infty\right]}(x) \end{aligned}$$

If $\gamma \in (0, 1)$ and

$$F^{\gamma=0}(x) = nx^{n-1} - (n-1)x^n$$

$$F^{\gamma=1}(x) = \frac{n^n}{(n-1)^n} x^n I_{\left[0, \frac{n-1}{n}\right]}(x) + I_{\left[\frac{n-1}{n}, \infty\right]}(x)$$

Then the expected auctioneer revenue is:

$$E(X^\gamma) = \frac{n-1}{n+1}, \quad \forall \gamma \in [0, 1]$$

If the auctioneer takes into account only expected revenue (risk-neutral auctioneer), then every auction mechanism belonging to *FSP* is equally attractive. However, every risk-averse auctioneer could prefer the auction mechanism with lowest variance (assuming, of course, that the bidders are risk-neutral).

The variance of the auctioneer revenue is:

$$V(X^\gamma) = \frac{(n-1)^2 (n(\gamma^2 - 2\gamma + 2) + 2(\gamma - 1))}{(n+2)(n+1)^2 (n+\gamma-1)^2}, \quad \forall \gamma \in [0, 1]$$

This decreases in γ and therefore the auction mechanism belonging to *FSP* with minimum revenue variance is the *FPA*.

The main problem with variance is that it does not indicate the sign of revenue deviations: revenue can be volatile and suddenly take high values. Given a confidence level β , the VaR for auctioneer revenue is the smallest number k such that the probability that the difference between expected and actual revenue (the loss) is less than k and greater than β in auction mechanism A^γ :

$$VaR_\beta^\gamma = \inf\{k \in [0, E(X^\gamma)]: P(E(X^\gamma) - X^\gamma < k) \geq \beta\}$$

$$= \inf\left\{k \in \left[0, \frac{n-1}{n+1}\right]: F^\gamma\left(\frac{n-1}{n+1} - k\right) \leq 1 - \beta\right\}.$$

Example

If there are two bidders ($n = 2$).

$$VaR_\beta^\gamma = \inf\left\{k \in \left[0, \frac{1}{3}\right]: F^\gamma\left(\frac{1}{3} - k\right) \leq 1 - \beta\right\}$$

$$= \begin{cases} \frac{3\sqrt{1-\gamma}\sqrt{\beta}-2+\gamma}{3(1+\gamma)} & \text{if } \gamma < 1-\beta \\ \frac{-3\sqrt{1-\beta}\sqrt{\gamma}+1+\gamma}{3(1+\gamma)} & \text{if } \gamma \geq 1-\beta \end{cases}$$

Figure 1 shows VaR for two bidders at a confidence level of 0.95 and $\gamma \in [0, 1]$.

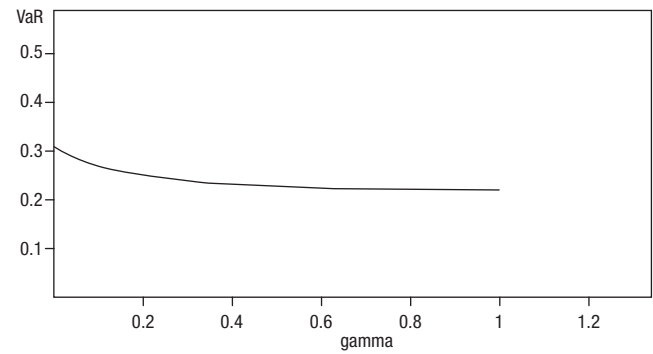


Figure 1 VaR_β^γ for $n = 2$.

2.2. Numerical results

In this section we give the VaR for auctioneer revenue for different numbers of bidders and different auction mechanisms of parameter γ . We take two typical confidence levels, 0.95 and 0.99. We also compute the relative VaR (RVaR) given by:

$$RVaR_\beta^\gamma = \frac{VaR_\beta^\gamma}{E(X^\gamma)} \times 100 = \frac{n+1}{n-1} VaR_\beta^\gamma \times 100.$$

Tables 1 and 2 list values of VaR and RVaR for different auction mechanisms and numbers of bidders for confidence levels of 0.95 and 0.99 respectively.

The auction mechanisms belonging to *FSP* with the lowest and highest VaR for revenue are *FPA* ($\gamma = 1$) and *SPA* ($\gamma = 0$) respectively. For example, if the number of bidders is $n = 4$, with probability ≥ 0.95 , the difference between expected

Table 1 $VaR_{0.95}^\gamma$ and $RVaR_{0.95}^\gamma$

γ/n	2	3	4	8	10	20	30	40	100
0	0.308 92.4%	0.3647 72.93%	0.3514 58.57%	0.2485 31.95%	0.2124 25.95%	0.1209 13.36%	0.0841 8.99%	0.0644 6.77%	0.0268 2.73%
0.1	0.2691 80.71%	0.3312 66.24%	0.324 54.01%	0.2332 29.98%	0.2 24.43%	0.1144 12.65%	0.0797 8.52%	0.0611 6.43%	0.02543 2.59%
0.2	0.25 75%	0.3041 60.82%	0.3007 50.11%	0.2194 28.21%	0.189 23.04%	0.1084 11.98%	0.0757 8.09%	0.0581 6.1%	0.0242 2.47%
0.3	0.2391 71.74%	0.2856 57.11%	0.2818 46.96%	0.2073 26.66%	0.1785 21.81%	0.103 11.39%	0.072 7.69%	0.0553 5.81%	0.023 2.35%
0.4	0.2323 69.7%	0.2738 54.76%	0.2682 44.7%	0.1971 25.35%	0.1699 20.76%	0.0983 10.86%	0.0687 7.35%	0.0528 5.55%	0.022 2.25%
0.5	0.2279 68.38%	0.2661 53.22%	0.2592 43.2%	0.1892 24.33%	0.163 19.92%	0.0943 10.42%	0.066 7.05%	0.0507 5.33%	0.0212 2.16%
0.6	0.2251 67.52%	0.261 52.2%	0.2532 42.2%	0.1836 23.6%	0.158 19.31%	0.0913 10.09%	0.0638 6.82%	0.049 5.15%	0.0205 2.09%
0.7	0.2233 66.99%	0.2577 51.54%	0.2493 41.55%	0.1799 23.13%	0.1546 18.9%	0.0891 9.85%	0.0623 6.66%	0.0478 5.03%	0.02 2.04%
0.8	0.2222 66.67%	0.2557 51.14%	0.2469 41.16%	0.1776 22.84%	0.1526 18.65%	0.0878 9.71%	0.0613 6.56%	0.0471 4.95%	0.0196 2%
0.9	0.2217 66.51%	0.2547 50.94%	0.2457 40.95%	0.1764 22.68%	0.1515 18.52%	0.0871 9.63%	0.0608 6.5%	0.0467 4.91%	0.0195 1.99%
1	0.2215 66.46%	0.2544 50.88%	0.2454 40.89%	0.1761 22.64%	0.1512 18.48%	0.087 9.61%	0.0607 6.49%	0.0466 4.9%	0.0194 1.98%

Table 2 $VaR_{0.99}^{\gamma}$ and $RV aR_{0.99}^{\gamma}$

γ/n	2	3	4	8	10	20	30	40	100
0	0.3283 98.5%	0.4411 88.22%	0.4591 76.52%	0.3677 47.28%	0.3225 39.42%	0.1936 21.39%	0.1371 14.65%	0.106 11.14%	0.0447 4.56%
0.1	0.3046 91.38%	0.4048 80.95%	0.4255 70.92%	0.3459 44.47%	0.3042 37.18%	0.1835 20.28%	0.1301 13.91%	0.1007 10.58%	0.0426 4.34%
0.2	0.2961 88.82%	0.3855 77.09%	0.4017 66.96%	0.3277 42.14%	0.2887 35.28%	0.1747 19.3%	0.124 13.26%	0.096 10.09%	0.0406 4.15%
0.3	0.2912 87.36%	0.3746 74.92%	0.3872 64.54%	0.3136 40.32%	0.2762 33.76%	0.1673 18.49%	0.1189 12.71%	0.092 9.67%	0.039 3.98%
0.4	0.2882 86.45%	0.3677 73.54%	0.3781 63.02%	0.3034 39.01%	0.2669 32.63%	0.1615 17.85%	0.1147 12.26%	0.0888 9.33%	0.0376 3.84%
0.5	0.2862 85.86%	0.3632 72.64%	0.3721 62.01%	0.2965 38.12%	0.2605 31.83%	0.1572 17.37%	0.1116 11.93%	0.0863 9.08%	0.0366 3.73%
0.6	0.2849 85.48%	0.3602 72.04%	0.3681 61.35%	0.2919 37.53%	0.2562 31.3%	0.1542 17.04%	0.1094 11.69%	0.0846 8.89%	0.0358 3.65%
0.7	0.2841 85.24%	0.3583 71.66%	0.3655 60.91%	0.2889 37.14%	0.2533 30.96%	0.1522 16.82%	0.1079 11.53%	0.0835 8.77%	0.0353 3.6%
0.8	0.2836 85.09%	0.3571 71.43%	0.3639 60.65%	0.287 36.9%	0.2515 30.74%	0.151 16.69%	0.107 11.44%	0.0827 8.7%	0.035 3.57%
0.9	0.2834 85.02%	0.3566 71.31%	0.3631 60.51%	0.286 36.78%	0.2506 30.63%	0.1503 16.62%	0.1065 11.39%	0.0824 8.66%	0.0348 3.55%
1	0.2833 85%	0.3564 71.27%	0.3628 60.47%	0.2857 36.74%	0.2503 30.6%	0.1502 16.6%	0.1064 11.37%	0.0823 8.65%	0.0348 3.54%

and actual revenue is ≤ 0.3514 in SPA and ≤ 0.2454 in FPA. For the same probability level, FPA yields smaller losses for the auctioneer. The expected revenue in this case is $3/5$, so at the 95% confidence level the loss will not exceed 58.57% in SPA nor exceed 40.89% in FPA.

The VaR for auctioneer revenue tends to 0 when the number of bidders increases in all auction mechanisms belonging to FSP, while revenue tends to 1.

By calculating VaR, an auctioneer can estimate at a given confidence level the maximum amount that can be lost, depending on the number of bidders and the auction mechanism chosen.

3. Other auction mechanisms

There are of course endless auction mechanisms. We analyzed a parametric family containing the two most common, but there are other auction mechanisms that while giving the same expected auctioneer revenue, can yield VaR that is less than that for FPA. This paper shows a method for measuring the risk and thus provides the auctioneer with a second criterion for choosing the most suitable auction mechanism.

Below we consider other auction mechanisms that can be found in different contexts: APA, SLA and SCA. The expected auctioneer revenue is the same as in any auction mechanism belonging to FSP. The VaR for auctioneer revenue for these other mechanisms was obtained by simulation.

3.1. All-pay auction

In this auction mechanism the highest bidder wins the object but all bidders pay their bid. The profit function for bidder $i \in \{1, \dots, n\}$ is:

$$B_i^{AP}(\theta_i, b_1, \dots, b_n) = \begin{cases} \theta_i - b_i & \text{if } b_i > \max_{j \neq i} b_j \\ -b_i & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

It is well known that the symmetric Bayesian Nash equilibrium $(b^{AP}(\theta_1), \dots, b^{AP}(\theta_n))$ is

$$b^{AP}(\theta_i) = \frac{(n-1)\theta_i^n}{n}.$$

Let X^{AP} be the random variable that gives the auctioneer revenue in APA:

$$X^{AP} = \frac{n-1}{n} \sum_{i=1}^n \Theta_i^n.$$

3.2. Sad-loser auction

In this auction mechanism, the highest bidder wins the object but pays nothing. All other bidders pay their bids. Then the profit function for bidder $i \in \{1, \dots, n\}$ is:

$$B_i^{SL}(\theta_i, b_1, \dots, b_n) = \begin{cases} \theta_i & \text{if } b_i > \max_{j \neq i} b_j \\ -b_i & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

It is well known that the symmetric Bayesian Nash equilibrium $(b^{SL}(\theta_1), \dots, b^{SL}(\theta_n))$ is

$$b^{SL}(\theta_i) = \frac{(n-1)\theta_i^n}{n(1-\theta_i^{n-1})}.$$

Hence let X^{SL} be the random variable that gives the auctioneer revenue in SLA:

$$X^{SL} = \frac{n-1}{n} \sum_{i=2}^n \frac{Y_{(i)}^n}{1 - Y_{(i)}^{n-1}},$$

where $Y_{(i)}$ is the i th-order statistic for $\{\Theta_1, \dots, \Theta_n\}$.

3.3. Santa Claus auction

Finally, in the SCA mechanism the highest bidder wins the object and pays his own bid b_i , $i \in \{1, \dots, n\}$, but all bidders receive the amount $\frac{b_i^n}{n}$. Then the profit function for bidder i is:

$$B_i^{SC}(\theta_i, b_1, \dots, b_n) = \begin{cases} \theta_i - b_i + \frac{b_i^n}{n} & \text{if } b_i > \max_{j \neq i} b_j \\ \frac{b_i^n}{n} & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

The symmetric Bayesian Nash equilibrium $(b^{SC}(\theta_1), \dots, b^{SC}(\theta_n))$ is given by

$$b^{SC}(\theta_i) = \theta_i.$$

Let X^{SC} be the random variable that gives the auctioneer revenue in SCA:

$$X^{SC} = Y - \frac{1}{n} \sum_{i=1}^n \Theta_i^n,$$

where Y is the highest-order statistic for $\{\Theta_1, \dots, \Theta_n\}$.

3.4. Numerical results

We calculated the sample VaR for the auctioneer revenue in the above three auction mechanisms. For this, each auction mechanism was repeated 100 000 times. Tables 3 and 4 show VaR and RVaR values for the auctioneer revenue for different numbers of bidders in SCA, APA and SLA mechanism at confidence levels of 0.95 and 0.99 respectively.

We can observe that the VaR for auctioneer revenue in SCA tends to 0. In SLA, the VaR for auctioneer revenue is

very high and decreases only slightly with increasing competition (for $n = 1000$, $\beta = 0.95$ the RVaR is 98.4%). In APA the VaR for auctioneer revenue is also very high, but seems to increase with competition (for $n = 1000$, $\beta = 0.95$ the RVaR is 91.25%), meaning the auctioneer can have lower VaR with two bidders than with another number of bidders. Menicucci (2009) proved that in APA, if the type distribution satisfies some specific assumptions, the volatility of auctioneer revenue increases with competition.

4. Conclusions

We analyzed a parametric family of auction mechanisms *FSP* containing FPA and SPA as particular cases. We assumed hypotheses that included independent and uniformly distributed $[0,1]$ types. We obtained a unique Bayesian Nash equilibrium for each auction mechanism belonging to *FSP*.

For a risk-neutral auctioneer, every auction mechanism belonging to *FSP* is equally attractive, but a risk-averse auctioneer would prefer the mechanism with the lowest VaR. We calculated the VaR for auctioneer revenue for the auction mechanisms belonging to *FSP*.

Calculation of VaR enables the auctioneer to estimate the maximum amount that can be lost at a given confidence level, depending on the number of bidders and the auction mechanism used.

The results could also be useful from another point of view: if an auctioneer knows in advance that the possible maximum loss is greater than what can be absorbed economically, a different auction mechanism can be chosen or the number of bidders can be increased to reduce this risk.

We must bear in mind that although the expected auctioneer revenue is a good long-term indicator, a sudden

Table 3 $VaR_{0.95}$ and $RVaR_{0.95}$

n	2	3	4	8	10	20	30	40	100
SCA	0.1909 57.26%	0.1931 38.62%	0.1798 29.96%	0.1217 15.65%	0.1007 12.31%	0.0571 6.31%	0.0391 4.18%	0.0297 3.13%	0.0123 1.25%
APA	0.3008 90.23%	0.4528 90.55%	0.5443 90.71%	0.7048 90.62%	0.7426 90.76%	0.822 90.85%	0.8493 90.79%	0.8653 90.96%	0.8916 90.96%
SLA	0.333 99.9%	0.498 99.6%	0.5961 99.35%	0.7697 98.96%	0.8086 98.83%	0.8926 98.65%	0.9215 98.51%	0.9374 98.55%	0.9649 98.44%

Table 4 $VaR_{0.99}$ and $RVaR_{0.99}$

n	2	3	4	8	10	20	30	40	100
SCA	0.2668 80.05%	0.3112 62.24%	0.2991 49.84%	0.2228 28.64%	0.1933 23.63%	0.1129 12.48%	0.0787 8.42%	0.0598 6.29%	0.025 2.55%
APA	0.3272 98.15%	0.4909 98.18%	0.5888 98.13%	0.764 98.23%	0.8032 98.17%	0.8886 98.21%	0.918 98.13%	0.9245 98.24%	0.9628 98.23%
SLA	0.3333 99.99%	0.4998 99.97%	0.5996 99.94%	0.7767 99.86%	0.8169 99.85%	0.903 99.81%	0.9333 99.76%	0.9493 99.79%	0.9781 99.78%

and unexpected decrease in actual revenue could still lead to bankruptcy. In addition, we considered other auction mechanisms (SLA, APA and SCA).

The variance and VaR followed the same order for the auctioneer revenue obtained in the different auction mechanisms considered. If $n = 2$, the order is $SCA < FPA < APA < SPA < SLA$. Table 5 summarizes the VaR and the RVaR values for all auction mechanisms for two bidders and confidence levels $\beta = 0.95$ and 0.99 . If $n > 2$, the order is $SCA < FPA < SPA < APA < SLA$. Tables 6 and 7 summarize the VaR and RVaR values for all auction mechanisms for different numbers of bidders and confidence levels $\beta = 0.95$ and 0.99 respectively.

We can observe that the VaR for auctioneer revenue in SCA tends to 0 when the competition increases, just like the auction mechanisms belonging to the family *FSP*. In addition, the VaR in SCA is lower than in any auction mechanism belonging to *FSP*.

SCA has the lowest VaR for auctioneer revenue among all the auction mechanisms analyzed and a VaR for bidder profit equal to 0. In SLA the VaR for auctioneer revenue is very high and only decreases slightly with increasing competition. In APA the VaR for auctioneer revenue is also very high, but seems to increase with competition. This means that the auctioneer can expect a lower VaR with two bidders than with any other number of bidders. Menicucci (2009) proved

Table 5 VaR_β and $RVaR_\beta$ for the revenue for $n = 2$, $\beta = 0.95, 0.99$

γ/β	0.95		0.99	
SLA	0.333	99.9%	0.3333	9.99%
0	0.308	92.4%	0.3283	98.5%
APA	0.3008	90.23%	0.3272	98.15%
0.1	0.2691	80.71%	0.3046	91.38%
0.2	0.25	75%	0.2961	88.82%
0.3	0.2391	71.74%	0.2912	87.36%
0.4	0.2323	69.7%	0.2882	86.45%
0.5	0.2279	68.38%	0.2862	85.86%
0.6	0.2251	67.52%	0.2849	85.48%
0.7	0.2233	66.99%	0.2841	85.24%
0.8	0.2222	66.67%	0.2836	85.09%
0.9	0.2217	66.51%	0.2834	85.02%
1	0.2215	66.46%	0.2833	85%
SCA	0.1909	57.26%	0.2668	80.05%

that if the type distribution satisfies some specific assumptions in APA, the volatility of auctioneer revenue increases with competition.

Table 6 $VaR_{0.95}$ and $RVaR_{0.95}$ for the revenue for $n > 2$

γ/n	3	4	8	10	20	30	40	100
SLA	0.498 99.6%	0.5961 99.35%	0.7697 98.96%	0.8086 98.83%	0.8926 98.65%	0.9215 98.51%	0.9374 98.55%	0.9649 98.44%
APA	0.4528 90.55%	0.5443 90.71%	0.7048 90.62%	0.7426 90.76%	0.822 90.85%	0.8493 90.79%	0.8653 90.96%	0.8916 90.96%
0	0.3647 72.93%	0.3514 58.57%	0.2485 31.95%	0.2124 25.95%	0.1209 13.36%	0.0841 8.99%	0.0644 6.77%	0.0268 2.73%
0.1	0.3312 66.24%	0.324 54.01%	0.2332 29.98%	0.2 24.43%	0.1144 12.65%	0.0797 8.52%	0.0611 6.43%	0.02543 2.59%
0.2	0.3041 60.82%	0.3007 50.11%	0.2194 28.21%	0.189 23.04%	0.1084 11.98%	0.0757 8.09%	0.0581 6.1%	0.0242 2.47%
0.3	0.2856 57.11%	0.2818 46.96%	0.2073 26.66%	0.1785 21.81%	0.103 11.39%	0.072 7.69%	0.0553 5.81%	0.023 2.35%
0.4	0.2738 54.76%	0.2682 44.7%	0.1971 25.35%	0.1699 20.76%	0.0983 10.86%	0.0687 7.35%	0.0528 5.55%	0.022 2.25%
0.5	0.2661 53.22%	0.2592 43.2%	0.1892 24.33%	0.163 19.92%	0.0943 10.42%	0.066 7.05%	0.0507 5.33%	0.0212 2.16%
0.6	0.261 52.2%	0.2532 42.2%	0.1836 23.6%	0.158 19.31%	0.0913 10.09%	0.0638 6.82%	0.049 5.15%	0.0205 2.09%
0.7	0.2577 51.54%	0.2493 41.55%	0.1799 23.13%	0.1546 18.9%	0.0891 9.85%	0.0623 6.66%	0.0478 5.03%	0.02 2.04%
0.8	0.2557 51.14%	0.2469 41.16%	0.1776 22.84%	0.1526 18.65%	0.0878 9.71%	0.0613 6.56%	0.0471 4.95%	0.0196 2%
0.9	0.2547 50.94%	0.2457 40.95%	0.1764 22.68%	0.1515 18.52%	0.0871 9.63%	0.0608 6.5%	0.0467 4.91%	0.0195 1.99%
1	0.2544 50.88%	0.2454 40.89%	0.1761 22.64%	0.1512 18.48%	0.087 9.61%	0.0607 6.49%	0.0466 4.9%	0.0194 1.98%
SCA	0.1931 38.62%	0.1798 29.96%	0.1217 15.65%	0.1007 12.31%	0.0571 6.31%	0.0391 4.18%	0.0297 3.13%	0.0123 1.25%

Table 7 $VaR_{0.99}$, $RVaR_{0.99}$ for the revenue for $n > 2$

γ/n	3	4	8	10	20	30	40	100
SLA	0.4998 99.97%	0.5996 99.94%	0.7767 99.86%	0.8169 99.85%	0.903 99.81%	0.9333 99.76%	0.9493 99.79%	0.9781 99.78%
APA	0.4909 98.18%	0.5888 98.13%	0.764 98.23%	0.8032 98.17%	0.8886 98.21%	0.918 98.13%	0.9245 98.24%	0.9628 98.23%
0	0.4411 88.22%	0.4591 76.52%	0.3677 47.28%	0.3225 39.42%	0.1936 21.39%	0.1371 14.65%	0.106 11.14%	0.0447 4.56%
0.1	0.4048 80.95%	0.4255 70.92%	0.3459 44.47%	0.3042 37.18%	0.1835 20.28%	0.1301 13.91%	0.1007 10.58%	0.0426 4.34%
0.2	0.3855 77.09%	0.4017 66.96%	0.3277 42.14%	0.2887 35.28%	0.1747 19.3%	0.124 13.26%	0.096 10.09%	0.0406 4.15%
0.3	0.3746 74.92%	0.3872 64.54%	0.3136 40.32%	0.2762 33.76%	0.1673 18.49%	0.1189 12.71%	0.092 9.67%	0.039 3.98%
0.4	0.3677 73.54%	0.3781 63.02%	0.3034 39.01%	0.2669 32.63%	0.1615 17.85%	0.1147 12.26%	0.0888 9.33%	0.0376 3.84%
0.5	0.3632 72.64%	0.3721 62.01%	0.2965 38.12%	0.2605 31.83%	0.1572 17.37%	0.1116 11.93%	0.0863 9.08%	0.0366 3.73%
0.6	0.3602 72.04%	0.3681 61.35%	0.2919 37.53%	0.2562 31.3%	0.1542 17.04%	0.1094 11.69%	0.0846 8.89%	0.0358 3.65%
0.7	0.3583 71.66%	0.3655 60.91%	0.2889 37.14%	0.2533 30.96%	0.1522 16.82%	0.1079 11.53%	0.0835 8.77%	0.0353 3.6%
0.8	0.3571 71.43%	0.3639 60.65%	0.287 36.9%	0.2515 30.74%	0.151 16.69%	0.107 11.44%	0.0827 8.7%	0.035 3.57%
0.9	0.3566 71.31%	0.3631 60.51%	0.286 36.78%	0.2506 30.63%	0.1503 16.62%	0.1065 11.39%	0.0824 8.66%	0.0348 3.55%
1	0.3564 71.27%	0.3628 60.47%	0.2857 36.74%	0.2503 30.6%	0.1502 16.6%	0.1064 11.37%	0.0823 8.65%	0.0348 3.54%
SCA	0.3112 62.24%	0.2991 49.84%	0.2228 28.64%	0.1933 23.63%	0.1129 12.48%	0.0787 8.42%	0.0598 6.29%	0.025 2.55%

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