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ARTICLE

Optimal just-in-time buffer inventory for preventive maintenance with imperfect quality items

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Abstract This paper deals with a just-in-time manufacturing environment which produces perfect quality items with defective items (a percentage of whole products) irrespective of the nature of the preventive maintenance. Since preventive maintenance is an essential part of production structure, performing of regular preventive maintenance results in a shutdown of the production unit for a period of time to enhance the condition of the production unit at an acceptable level. During the shutdown period just-in-time buffers for both the perfect and imperfect quality items are needed to continue the normal operation. The period of preventive maintenance depends on the nature and condition of the production unit which is random in nature. The percentage of imperfect quality item is also random. The optimal just-in-time buffer is determined to minimize the system running cost by considering the holding cost of perfect and imperfect quality items and shortage cost of perfect and imperfect quality items. A numerical example is presented to illustrate the development of the model and sensitivity of the model is analyzed.

1. Introduction

After the introduction of just-in-time production system in the literature of classical inventory, it has been widely accepted due to considerable reduction in material inventories. This reduction led some people to adopt the wrong notation that inventory should be totally eliminated. But, some inventories are required to operate the production system efficiently in the case of preventive maintenance. Maintenance is an integral part of business operations that spans the whole spectrum of activities from acquisition to retirement of production unit and its equipment. Moreover,

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effective and efficient maintenance affects asset optimization by providing equipment reliability and aims to improve service to customers whilst reducing crises of production. Maintenance contributes to the profitability of the process mainly by keeping the system functioning and capable of fulfilling production needs for longer period of time by providing higher system availability. It is also well known that effective maintenance strategy efficiently controls capacity of utilization. Olorunniwo and Izuchkwu's (1987) noted that preventive maintenance has been entirely based on one of the two extreme assumptions: the production unit is enhanced to either a good-as-new or bad-as-old condition after maintenance. According to British Standard Institute (1974), maintenance is nothing but the combination of action taken to restore an item to retain it in an acceptable condition. However, one of the basic problems for people working with polymer, cotton, leather industries, etc. is that it is quite impossible to produce 100% perfect quality items. Due to some uncontrollable factors of the production unit, some items are imperfect or not to the standard maintained by the manufacturing unit. This behavior inspired many researchers to work on imperfect quality items in details. Cheng (1991) formulated an economic order quantity model with demand dependent unit production cost and imperfect production process. Zhang and Gerchak (1990) proposed a joint lot sizing and inspection policy under an EOQ (economic order quantity) model where a random proportion of units is defective and the defective units cannot be used and they must be replaced by non-defective ones. Schwaller (1988) presented a procedure that extended EOQ models by adding the assumptions that defective items of a known proportion were present in incoming lots and that fixed and variable inspection costs were incurred in finding and removing the items. Cardenas-Barron (2009) investigated an EPQ (economic production quantity) model with reworking process at a single stage production system applying planned backorders. Cardenas-Barron, Smith, and Goyal (2010) determined the optimal ordering policies for a buyer who operated an inventory policy with planned backorders while the supplier offered a temporary fixed-percentage discount. Cardenas-Barron, Sarkar, and Trevino-Garza (2013) investigated both the optimal replenishment lot size and the optimal number of shipments jointly. Sarkar, Cardenas-Barron, Sarkar, and Singgih (2014) revisited the EPQ (economic production quantity) model with rework process at a single-stage manufacturing system with planned backorders. Sarkar and Saren (2016) studied a deterioration production system for an imperfect production system with inspection errors and warranty cost in which the in-control state shifted randomly from the out-of-control. The effect of imperfect quality items on optimal order quantity and total running cost of the system is noted in the works of Rosenblatt and Lee (1986), Chakravarty and Shtub (1987), Urban (1992), Anily (1995), Salameh and Jaber (2000), Sana (2010a, 2010b, 2012), DasRoy, Sana, and Chaudhuri (2012), etc.

In this literature, plenty of articles are available under the basic assumption that, after preventive maintenance, the production system begins with the in-control state which is shifted to the out-of-control state and later produces nonconforming items. Due to imperfect repair during preventive maintenance, the system fails to operate and performing minimal repair and the production can be started again within minimum span of time. Groenvelt, Pmtelon, and Seidmann (1992a) and Groenvelt, Pmtelon, and Seidmann (1992b) focused the problem of determining the economic lot size for an unreliable manufacturing facility and showed a trade off to exist between the overall investment to increase the maintenance level that results saving in safety stock and repair cost. Van Der Duyn Schouten and Vanneste (1997) proposed a preventive maintenance policy which was based on the information about the age of the installation and the inventory buffer. Balasubramanian (1987) proposed an approach for preventive maintenance scheduling in the light of production plan. To cope with these situations, several strategies were proposed those were found in the articles of Rosenblatt and Lee (1986), Porteus (1986), Hariga and Ben-Daya (1998), Lee and Rosenblatt (1989), Cardenas-Barron (2000), Goyal and Cardenas-Barron (2002), Panda (2007), etc.

The proposed model considers that the regular preventive maintenance improves the condition of the production unit to an acceptable level that prevents sudden failure and maintains the quality of the original as new as one. During the preventive maintenance, a just-in-time buffer inventory is needed to maintain the normal operation. Since the production unit produces both of perfect and imperfect quality items, there is a market for imperfect quality items then just-in-time buffer inventory is needed for imperfect quality items also to maintain the normal operation. As the system produces imperfect quality items in a random percentage due to some uncontrollable factors of production after preventive maintenance, not for imperfect repair during preventive maintenance, we consider the buffer inventory of the perfect quality products to minimize the total system running cost. Mainly, we discuss the effects of imperfect quality items on the buffer inventory of perfect quality and hence on the system running cost. The rest of the paper is organized as follows. In the next section, the assumptions and notations for the development of the model are proposed. The mathematical model is developed in Section 3. Sections 4 and 5 provide the numerical illustration and concluding remarks, respectively.

2. Assumptions and notations

The following assumptions and notations are used to develop the model:

- D_1 and D_2 are the consumption rate of perfect and imperfect quality items per unit time, respectively.
- Number of defective items is in percentage p which is assumed to be a random variable having probability density function $f_1(p)$. The term p' is the maximum percentage of imperfect quality items.
- Screening time to differentiate perfect and imperfect quality items is negligible. Screening cost is also negligible.
- System running time T is large in comparison to the preventive maintenance time t so that, during any time

period T, buffer replenishment of perfect quality item starts from zero level.

- The regular preventive maintenance ensures that the probability of breakdown of the production unit during T is negligible, i.e., approximately zero.
- Before the starting of any normal preventive maintenance, the just-in-time buffer inventory for perfect quality inventory is Q_1.
- Unused buffer inventory during t is depleted to zero during the next cycle time T.
- Regular preventive maintenance time t is a random variable having the probability density function $f_2(p)$.
- The buffer replenishment rate for perfect quality inventory is k_1 unit per unit time.
- h_1 and h_2 are the holding cost per unit per unit time for perfect and imperfect quality items, respectively. S_p and S_i are the shortage cost per unit for perfect and imperfect quality items, respectively.
- $X(p, t)$ is the joint density function of p and t.

3. The mathematical model

Under the just-in-time structure, the production unit produces both perfect and imperfect quality items. At the beginning of the preventive maintenance cycle to fulfill the demand of perfect quality items, the system produces perfect quality at rate D_1 units per unit time. Since p% of produced items are defective, thus for the production of D_1 units of perfect quality $pD_1/(1-p)$ units of imperfect quality items are produced. Toward the end of normal operation time T, the buffer inventory for perfect quality raises to Q_1 at a rate k_1 units per unit time. Thus total amount of buffer inventory for imperfect quality is $pQ_1/(1-p)$, which is replenished at a rate $pk_1/(1-p)$, units per unit time. However this finite replenishment starts Q_1/k_1 units time before the end of the time period T. At the end of time period T, the regular preventive maintenance starts and continues for a period of time t. The total system running cost involves holding cost and shortage cost for perfect and imperfect quality items (if we assume that the maintenance cost is fixed). We now calculate the expected unit total cost for perfect and imperfect quality items separately.

3.1. Expected unit total cost for perfect quality inventory

From the beginning of the production cycle up to the time $(T-Q_1/k_1)$, the perfect quality items are consumed immediately. The inventory is kept in stock for (Q_1/k_1) units of time in buffer through the replenishment at a rate k_1 unit per unit time. Then, it is depleted during preventive maintenance. The behavior of inventory level is depicted in Fig. 1.

The average amount of perfect quality inventory during the time period $(T+t)$ is

$$\left(\frac{1}{2}Q_1\frac{Q_1}{k_1}+\frac{1}{2}Q_1\frac{Q_1}{D_1}\right)\left(\frac{1}{T+t}\right)=\left(\frac{k_1+D_1}{k_1D_1}\right)\left(\frac{Q_1^2}{T+t}\right)$$

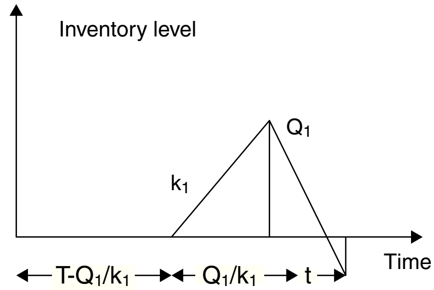

Figure 1 Behavior of perfect quality inventory.

If the buffer supply time (Q_1/D_1) for perfect quality inventory is less than the preventive maintenance time t then shortages occur and the stock out time is $(t-Q_1/D_1)$, otherwise stock out time is zero. Thus average shortage of perfect quality inventory per preventive maintenance cycle can be found as

$$S_1=\begin{cases}0 & \text{if} \quad t<Q_1/D_1\\ D_1(t-Q_1/D_1) & \text{if} \quad t>Q_1/D_1\end{cases}$$

Therefore, the expected unit total cost for perfect quality inventory per preventive maintenance cycle is given by

$$E\pi_p(Q_1)=h_1\int_0^{p'}\int_0^{\infty}\left(\frac{k_1+D_1}{k_1D_1}Q_1^2\right)\left(\frac{X(p,t)}{T+t}\right)dtdp + S_p\int_{Q_1/D_1}^{p'}\int_0^{\infty}D_1\left(t-\frac{Q_1}{D_1}\right)\left(\frac{X(p,t)}{T+t}\right)dtdp \quad (1)$$

3.2. Expected unit total cost for imperfect quality inventory

From the beginning of the production cycle, the demand of the imperfect quality is D_2 units per unit time. Due to the production of D_1 units of perfect quality items, amount of produced imperfect quality items is $pD_1/(1-p)$ units per unit time and it continues up to the time $(T-Q_1/k_1)$. In the time interval $\left[\frac{T-Q_1}{k_1}, T\right]$ for the increment of buffer stock of perfect quality inventory at a rate k_1 units per unit time, $pk_1/(1-p)$ units of imperfect quality items are produced per unit time up to the end of cycle time T. Now, the demand production relationship of imperfect quality items can be completely characterized by analyzing the following three cases:

Case 1: $D_2<\left(\frac{pD_1}{1-p}\right)<\left(\frac{p(D_1+k_1)}{1-p}\right)$

Case 2: $\left(\frac{pD_1}{1-p}\right)<D_2<\left(\frac{p(D_1+k_1)}{1-p}\right)$

Case 3: $\left(\frac{pD_1}{1-p}\right)<\left(\frac{p(D_1+k_1)}{1-p}\right)<D_2$

3.2.1. Case-1: When $D_2<\left(\frac{pD_1}{1-p}\right)<\left(\frac{p(D_1+k_1)}{1-p}\right)$

In this case the demand of imperfect quality items D_2 is less than the amount of imperfect quality items $pD_1/(1-p)$.

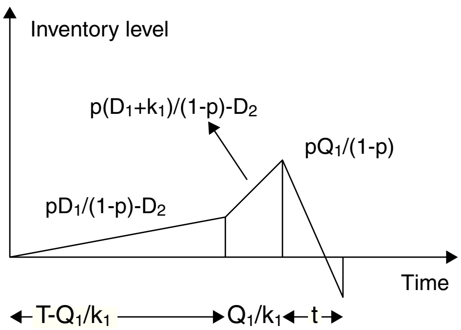

Figure 2 Behavior of imperfect quality inventory (Case 1).

Thus from the beginning of the production cycle after fulfilling the demand, the imperfect quality inventory level raises at a rate $(pD_1/(1-p)-D_2)$ units per unit time up to the point of time $(T-Q_1/k_1)$. Then, for the buffer stock additional k_1 units of perfect quality, inventory is produced per unit time and imperfect quality items are produced at a rate $[p(D_1+k_1)/(1-p)]$ per unit time for the time Q_1/k_1. Thus during the time period $\left[\frac{T-Q_1}{k_1}, T\right]$, the inventory level raises at a rate $\left(\frac{p(D_1+k_1)}{1-p}-D_2\right)$ units per unit time. Here, the logistics diagram of inventory level is depicted in Fig. 2.

Total amount of inventory at the end of production cycle before the starting of preventive maintenance is $\left(\left[\frac{pD_1}{1-p}-D_2\right]T+Q_1\right)$. During the preventive maintenance, it is depleted at a rate D_2 units per unit time. Therefore, the average amount of imperfect quality items for the entire maintenance cycle is

$$H_{i1}=\frac{1}{T+t}\left[\frac{1}{2}\left(\frac{pD_1}{1-p}-D_2\right)T^2+\frac{1}{2}\left(\frac{pQ_1}{1-p}\right)\left(\frac{Q_1}{k_1}\right)+\frac{1}{2}\frac{\left[\left(\frac{pD_1}{1-p}-D_2\right)T+\frac{pQ_1}{1-p}\right]^2}{D_2}\right]$$

$$=\frac{1}{2(T+t)}\left[\left(\frac{p(D_1+D_2)-D_2}{1-p}\right)T^2+\left(\frac{pQ_1^2}{(1-p)k_1}\right)+\frac{1}{D_2}\left[\left(\frac{p(D_1+D_2)-D_2}{1-p}\right)T+\frac{pQ_1}{1-p}\right]^2\right]$$

If the preventive maintenance time $t>\frac{1}{D_2}\left[\left(\frac{pD_1}{1-p}-D_2\right)T+\frac{pQ_1}{1-p}\right]=t_1$, then shortages occur otherwise there is no shortage. Thus average shortage is

$$S_{i1}=\begin{cases}0 & \text{if} \quad t\le t_1\\ D_2(t-t_1) & \text{if} \quad t>t_1\end{cases}$$

Hence, the expected unit total cost for imperfect quality inventory for the entire cycle in this case is

$$E\pi_{i1}(Q_1)=\frac{h_2}{2}\int_{\frac{D_2}{D_1+D_2}}^{p'}\int_0^{\infty}\left[\left(\frac{p(D_1+D_2)-D_2}{1-p}\right)T^2+\left(\frac{pQ_1^2}{(1-p)k_1}\right)\right.$$
$$\left.+\frac{1}{D_2}\left[\left(\frac{p(D_1+D_2)-D_2}{1-p}\right)T+\frac{pQ_1}{1-p}\right]^2\right]T\frac{X(p,t)}{T+t}dtdp+S_i\int_{\frac{D_2}{D_1+D_2}}^{p'}\int_{t_1}^{\infty}D_2(t-t_1)\frac{X(p,t)}{T+t}dtdp \qquad (2)$$

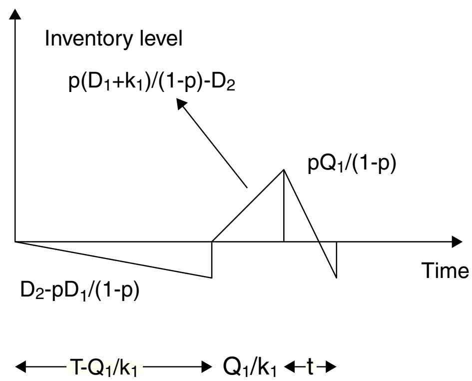

Figure 3 Behavior of imperfect quality inventory (Case 2).

3.2.2. Case-2: When $\left(\frac{pD_1}{1-p}\right)<D_2<\left(\frac{p(D_1+k_1)}{1-p}\right)$ holds

The demand of imperfect quality items, in this case, exceeds the amount of produced imperfect quality Items $\left(\frac{pD_1}{1-p}\right)$ per unit time. Thus there is shortage for imperfect quality items which is accumulated at a rate $\left(D_2-\frac{pD_1}{1-p}\right)$ per unit time upto the time $(T-Q_1/k_1)$. After that for the production of buffer inventory, imperfect quality items are produced at a rate $\left(\frac{p(D_1+k_1)}{1-p}\right)$ per unit time and $D_2<\frac{p(D_1+k_1)}{1-p}$. Therefore, inventory level of imperfect quality items raises at a rate $\left(D_2-\frac{p(D_1+k_1)}{1-p}\right)$ units per unit time, after fulfilling the demand in the time interval $(0, T-Q_1/k_1)$. At the end of production cycle, total amount of buffer inventory for imperfect quality items is $\left[\frac{p}{1-p}(D_1+k_1)-D_2\right]\frac{Q_1}{k_1}$. This amount of inventory depletes at a rate D_2 units per unit time during the preventive maintenance. The behavior of inventory level is depicted in Fig. 3. Therefore, the average amount of inventory is

$$H_{i2} = h_2\left[\frac{1}{2}\left(\frac{p}{1-p}(D_1+D_2)-D_2\right)\left(\frac{Q_1}{k_1}\right)^2 + \frac{1}{2}\left(\frac{p}{1-p}(D_1+D_2)-D_2\right)\frac{\frac{p}{1-p}(D_1+D_2)-D_2}{D_2}\left(\frac{Q_1}{k_1}\right)^2\right]\frac{1}{(T+t)}$$
$$= \frac{1}{(T+t)}\left[(p(D_1+D_2+k_1)-D_2)Q_1^2\frac{p(D_1+D_2+k_1)}{D_2}\right]$$

And, the average amount of shortage is

$$S_{i2} = \begin{cases} \left(\frac{s_i}{T+t}\right)\left(\frac{D_2-p(D_1+D_2)}{2(1-p)}\right)\left(T-\frac{Q_1}{k_1}\right)^2 & \text{if} \quad t \le t_2 \\ \left(\frac{s_i}{T+t}\right)\left(\frac{D_2-p(D_1+D_2)}{2(1-p)}\right)\left(T-\frac{Q_1}{k_1}\right)^2 + D_2(t-t_2) & \text{if} \quad t > t_2 \end{cases}$$

where t_2 is given by

$$t_2 = \frac{(p(D_1+D_2+k_1)-D_2)}{D_2}Q_1$$

Hence, the expected unit total cost is given by

$$E\pi_{i2}(Q_1) = \frac{h_2}{2k_1^2}\int_{\frac{D_2}{D_1+D_2+k_1}}^{\frac{D_2}{D_1+D_2}}\int_0^{\infty} p(D_1+D_2+k_1) - D_2]Q_1^2\frac{p(D_1+D_2+k_1)}{D_2}\frac{X(p,t)}{T+t}dtdp$$
$$+ S_i\int_{\frac{D_2}{D_1+D_2+k_1}}^{\frac{D_2}{D_1+D_2}}\int_0^{\infty}\left(\frac{D_2-p(D_1+D_2)}{2(1-p)}\right) \times \left(T-\frac{Q_1}{k_1}\right)^2\frac{X(p,t)}{T+t}dtdp$$
$$+ S_i\int_{\frac{D_2}{D_1+D_2+k_1}}^{\frac{D_2}{D_1+D_2}}\int_{t_2}^{\infty} D_2(t-t_2)\frac{X(p,t)}{T+t}dtdp \qquad (3)$$

3.2.3. Case-3: When $\left(\frac{pD_1}{1-p}\right) < \left(\frac{p(D_1+k_1)}{1-p}\right) < D_2$ holds

In this case, shortages occur for the entire preventive maintenance cycle. Therefore, the production of imperfect quality items during the time interval $(0,\ T-Q_1/k_1)$ as well as in the time interval $\left(T-\frac{Q_1}{k_1},T\right)$ are less than the demand of imperfect quality items D_2. The behavior of inventory level is depicted in Fig. 4. Thus the holding cost is zero throughout the maintenance cycle. And, the average amount of shortage is

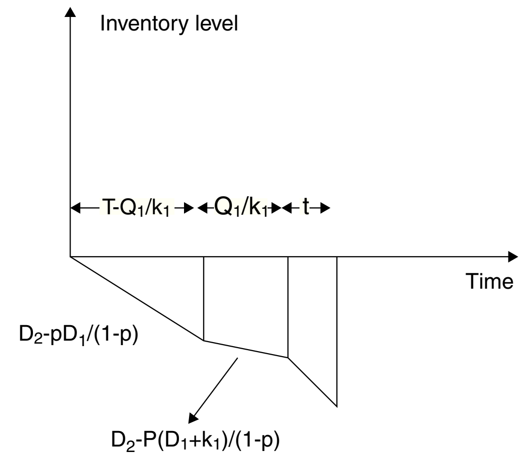

Figure 4 Behavior of imperfect quality inventory (Case 3).

$$S_{i3} = \frac{1}{2}\left(D_2-\frac{p}{1-p}D_1\right)\left(T-\frac{Q_1}{k_1}\right)^2 + \frac{1}{2}\left(D_2-\frac{p}{1-p}(D_1+k_1)\right)\left(\frac{Q_1}{k_1}\right)^2$$
$$+ D_2t = \frac{1}{2}\left(\frac{D_2-p(D_1+D_2)}{1-p}\right)\left(T-\frac{Q_1}{k_1}\right)^2 + \frac{1}{2}\left(\frac{D_2-p(D_1+D_2+k_1)}{1-p}\right)\left(\frac{Q_1}{k_1}\right)^2 + D_2t$$

Therefore, the expected unit total cost is given by

$$E\pi_{i3}(Q_1) = S_i\int_{\frac{D_2}{D_1+D_2+k_1}}^{\frac{D_2}{D_1+D_2}}\int_0^{\infty}\left[\frac{D_2-p(D_1+D_2)}{2(1-p)}\left(T-\frac{Q_1}{k_1}\right)^2 + \frac{D_2-p(D_1+D_2+k_1)}{2(1-p)}\left(\frac{Q_1}{k_1}\right)^2 + D_2t\right]\left(\frac{X(p,t)}{(T+t)}\right)dtdp \qquad (4)$$

3.3. Expected unit total cost for the system

The expected unit total cost of the system can be found as the sum of expected average holding cost and shortage cost of perfect quality and imperfect quality items, respectively, and is given by where

$$E\pi(Q_1) = E\pi_p(Q_1) + E\pi_{i1}(Q_1) = E\pi_1(Q_1) \quad \text{if} \quad D_2 < E\left(\frac{pD_1}{1-p}\right) < E\left(\frac{p(D_1+k_1)}{1-p}\right) \qquad (5)$$

$$E\pi(Q_1) = E\pi_p(Q_1) + E\pi_{i2}(Q_1) = E\pi_2(Q_1) \quad \text{if} \quad E\left(\frac{pD_1}{1-p}\right) < D_2 < E\left(\frac{p(D_1+k_1)}{1-p}\right) \qquad (6)$$

$$E\pi(Q_1) = E\pi_p(Q_1) + E\pi_{i3}(Q_1) = E\pi_3(Q_1) \quad \text{if} \quad E\left(\frac{pD_1}{1-p}\right) < E\left(\frac{p(D_1+k_1)}{1-p}\right) < D_2 \qquad (7)$$

$E\pi_p(Q_1)$, $E\pi_{i1}(Q_1)$, $E\pi_{i2}(Q_1)$ and $E\pi_{i3}(Q_1)$ are given by Eqs. (1)–(4), respectively. It should be noted that $X(p,\ t)$ is the joint density function of the percentage of defective items p and preventive maintenance time t. Since there is no inter-relationship between p and t, therefore, these two random variables are completely independent of each other and their joint density function can be expressed as the product of their individual density functions, i.e., $X(p, t) = f_1(p)f_2(t)$. Our problem is to determine Q_1 which minimizes $E\pi(Q_1)$. The necessary condition for this is $dE\pi(Q_1)/dQ_1 = 0$. This yields after simplification as follows:

$$h_1\frac{k_1+D_1}{k_1D_1}Q_1E\left(\frac{1}{T+t}\right)-S_p\int_{\frac{Q_1}{D_1}}^{\infty}\left(\frac{f_2(t)}{T+t}\right)dt+h_2E\left(\frac{1}{T+t}\right)\int_{\frac{D_2}{D_1+D_2}}^{p'}\frac{p}{1-p}\left(\frac{1}{k_1}+\frac{p}{D_2(1-p)}\right)f_1(p)dp$$
$$-S_i\int_{\frac{D_2}{D_1+D_2}}^{p'}\int_{t_1}^{\infty}\frac{p}{1-p}\frac{f_1(p)f_2(t)}{T+t}dtdp=0 \quad \text{if} \quad D_2<E\left(\frac{pD_1}{1-p}\right)<E\left(\frac{p(D_1+k_1)}{1-p}\right) \tag{8}$$

$$h_1\frac{k_1+D_1}{k_1D_1}Q_1E\left(\frac{1}{T+t}\right)-S_p\int_{\frac{Q_1}{D_1}}^{\infty}\left(\frac{f_2(t)}{T+t}\right)dt+\frac{h_2}{k_1^2}E\left(\frac{1}{T+t}\right)\int_{\frac{D_2}{D_1+D_2+k_1}}^{\frac{D_2}{D_1+D_2}}\frac{p(D_1+D_2+k_1)}{D_2}[p(D_1+D_2+k_1)-D_2]Q_1f_1(p)dp$$
$$-\frac{S_i}{k_1}\int_{\frac{D_2}{D_1+D_2+k_1}}^{\frac{D_2}{D_1+D_2}}\left[E\left(\frac{1}{T+t}\right)\left(\frac{D_2-p(D_1+D_2)}{2(1-p)}\right)\left(T-\frac{Q_1}{k_1}\right)+k_1\int_{t_2}^{\infty}\frac{p(D_1+D_2+k_1)-D_2}{1-p}\frac{f_2(t)}{T+t}dt\right]f_1(p)dp=0$$
$$\text{if} \quad E\left(\frac{pD_1}{1-p}\right)<D_2<E\left(\frac{p(D_1+k_1)}{1-p}\right) \tag{9}$$

and

$$h_1\frac{k_1+D_1}{k_1D_1}Q_1E\left(\frac{1}{T+t}\right)-S_p\int_{\frac{Q_1}{D_1}}^{\infty}\left(\frac{f_2(t)}{T+t}\right)dt$$
$$+S_iE\left(\frac{1}{T+t}\right)\int_0^{\frac{D_2}{D_1+D_2+k_1}}\left[2\frac{D_2-p(D_1+D_2)}{k_1^2(1-p)}-\frac{p}{k_1(1-p)}\right]f_1(p)dp=0 \quad \text{if} \quad E\left(\frac{pD_1}{1-p}\right)<E\left(\frac{p(D_1+k_1)}{1-p}\right)<D_2 \tag{10}$$

To evaluate the nature of the expected unit cost function $E\pi(Q_1)$, it is necessary to ascertain when the function is convex. But, it is difficult to verify that $d^2E\pi(Q_1)/dQ_1^2$ is greater than zero to draw the conclusion about its convexity directly. Thus, an indirect approach is applied to verify the convexity of $E\pi(Q_1)$. A parametric study is carried out over the expected unit cost function for several values of Q_1, and the response indicates that $E\pi(Q_1)$ is convex. However, it is difficult to derive a closed form solution for Q_1. Only numerical solution can be obtained by using suitable numerical method. From the input values of the parameters, we have to calculate $E\left(\frac{pD_1}{1-p}\right)$ and $E\left(\frac{p(D_1+k_1)}{1-p}\right)$. If $D_2 < E\left(\frac{pD_1}{1-p}\right)$ solve Eq. (8) and substitute the optimal value of Q_1 in (5) to get expected minimum unit system cost. If $E\left(\frac{p(D_1+k_1)}{1-p}\right) < D_2$, substitute the expected value of Q_1 found by solving Eq. (10) in (7) to obtain the expected unit system cost. Otherwise, solve Eq. (9) and find the expected unit system running cost from Eq. (6). Here, we use Newton–Raphson method to solve Eqs. (8)–(10).

3.4. Buffer inventory with variable buffer replenishment rate

Three different cost functions are derived above under three different scenarios arise for the control of imperfect quality items considering fixed buffer replenishment rate. In case-1, $\left(D_2 < E\left(\frac{pD_1}{1-p}\right) < E\left(\frac{p(D_1+k_1)}{1-p}\right)\right)$, the just-in-time configuration for the perfect quality items is maintained but this situation for imperfect quality items is violated. Since $D_2 < E\left(\frac{pD_1}{1-p}\right)$, some units of imperfect quality items are kept in stock, whose amount increases with time upto the point of time T. Then, it is depleted due to its demand during preventive maintenance. The just-in-time structure for imperfect quality items can be preserved only when the per unit time production of perfect quality items is lower than D_1. But, D_1 is fixed and lower volume of D_1 is not intended to any decision maker. In addition, the rate of production of imperfect quality is totally random. Hence, the JIT structure for imperfect quality items can never be maintained. The holding cost incurred in $[0, T-Q_1/k_1]$ is not controllable by the decision maker. During the building of buffer stock of perfect quality items, some imperfect quality items are also produced. Consequently, some amount of holding costs for both perfect quality items and imperfect quality items are incurred on the total expected unit system running cost. A significant amount of this cost may be reduced through suitable determination of the rate of buffer replenishment beyond its capacity. The buffer replenishment starts after $(T-Q_1/k_1)$ units of time and the holding cost incurred in the time interval $[0, T-Q_1/k_1]$ for imperfect quality items only. The holding costs for both perfect and imperfect quality items are incurred for (Q_1/k_1) units of time. Thus, if the buffer replenishment rate k_1 increases, then the average amount of perfect and imperfect quality items during buffer replenishment decrease and holding cost and the expected unit system cost decrease simultaneously, and the length of the time interval $[0, T-Q_1/k_1]$ increases. During this time, the holding cost for imperfect quality items increases. Since the holding cost per unit per unit time of imperfect quality items is less than that of perfect quality items, the increment of average amount of imperfect quality items due to the increment of the length of the time interval $[0, Q_1/k_1]$ is lower than that of perfect and imperfect quality items during buffer replenishment. In this situation, we predict that the lower buffer replenishment rate always leads to higher system cost. By natural selection, the system will try always to attain maximum buffer replenishment capacity. If $K(> (D_1 + k_1) > D_1)$ be the maximum capacity of production of perfect quality items per unit time of the system, then our problem is to determine Q_1 and k_1 simultaneously for the minimization of expected unit total cost of the system. This leads to the following constraint optimization problem as follows:

$$\text{Minimize} \quad E\pi_1(Q_1, k_1) \tag{11}$$

$$\text{subject to} \quad k_1 < K - D_1$$

$$Q_1, k_1 \geq 0$$

However, in the decision making situation, verification is needed whether the increment of buffer replenishment rate beyond its capacity leads to diminish the expected unit system running cost or not. In case-2, shortage of imperfect quality items are continued from the beginning of the maintenance cycle upto the point of time $(T - Q_1/k_1)$ and the imperfect quality items are stored in buffer from $(T - Q_1/k_1)$ to T. By the same logic, for shortage instead of holding imperfect quality inventory, as in case-1, we can conclude that the shortage of imperfect quality items during $[0, T - Q_1/k_1]$ cannot be avoided because it is uncontrollable. Only the holding cost of perfect and imperfect quality items may be reduced by adjusting the rate of production k_1 of perfect quality items. Then, from the problem, we have as follows:

$$\text{Minimize} \quad E\pi_2(Q_1, k_1) \tag{12}$$

$$\text{subject to} \quad k_1 \leq K - D_1$$

$$Q_1, k_1 \geq 0$$

we have to determine Q_1 and k_1 and compare the cost with the fixed buffer replenishment rate. In case-3, shortage of imperfect quality items is accumulated throughout the preventive maintenance cycle because $E\left(\frac{pD_1}{1-p}\right) < E\left(\frac{p(D_1+k_1)}{1-p}\right) < D_2$. Shortage of any product, whether it is perfect or imperfect, for the entire preventive maintenance cycle is not desired to any decision maker for the loss of goodwill as well as lost sale and the violation of JIT configuration. The shortage of imperfect quality items upto the point of time $(T - Q_1/k_1)$ is unavoidable because the decision maker has no control over it but it can be avoided or reduced further by suitably adjusting the buffer replenishment rate beyond its capacity. Clearly, two situations appear: (i) k_1 is adjusted beyond its capacity in such a fashion that $D_2 < E\left(\frac{p(D_1+k_1)}{1-p}\right)$, or (ii) the demand D_2 is so high that the adjustment exceeds the capacity of k_1. The case (ii) reveals that the demand of imperfect quality items may be close or higher than D_1 which is unrealistic to some extent. If we consider the first case, then it will be converted to case-2 and we have to determine Q_1 and k_1 from the following non-linear constraint optimization problem:

$$\text{Minimize} \quad E\pi_2(Q_1, k_1) \tag{13}$$

$$\text{subject to} \quad k_1 \leq K - D_1$$

$$D_2 \leq E\left(\frac{p(D_1 + k_1)}{1 - p}\right)$$

$$Q_1, k_1 \geq 0$$

However, it is worth mentioning that if the expected unit total system cost obtained by adjusting the buffer replenishment rate of perfect quality items, k_1, is higher than that obtained for the fixed buffer replenishment rate, then, depending on the priority of preference, the decision maker will have to decide between minimum cost and loss of goodwill which will be selected. Otherwise, he will prefer to adjust the buffer replenishment rate. In the next section, we present a numerical example which may give some idea of these conflicting situations under decision making environment. The constraint optimization problems (11)–(13) are solved by using penalty function method.

4. Numerical illustration

To illustrate the proposed model, we consider a numerical example in which the parameter values are taken as $T = 30$ days, $D_1 = 500$ units/day, $h_1 = \$0.4$, $h_2 = \$0.1$, $S_p = \$6$, $S_i = \$3$. The percentage of imperfect quality items follows uniform distribution with the probability density function

$$f_1(p) = \begin{cases} \dfrac{1}{0.1 - 0} & \text{if} \quad 0 \leq p \leq 0.1 \\ 0 & \text{otherwise} \end{cases}$$

The preventive maintenance time also follows uniform distribution with probability density function

$$f_2(t) = \begin{cases} \dfrac{1}{4 - 0.5} & \text{if} \quad 0.5 \leq t \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

For fixed buffer replenishment rate, $k_1 = 100$, the optimal buffer inventory for perfect quality items and corresponding expected unit total cost of the system are provided in Table 1. $E\pi^*(0)$ indicates the expected unit total cost of the

Table 1 Optimal values of Q_1 and $E\pi$ for Q_1^* and no buffer for fixed buffer replenishment rate ($k_1 = 100$).

D_2	Q_1^*	$E\pi(Q_1^*)$	$Q_1 = k_1$	$E\pi^*(0)$	% increase in $E\pi^*$ for no buffer	Remarks
20	712.598	261.273	7.1259	328.385	25.686	Case-1
25	750.891	197.44	7.5089	271.767	37.645	Case-1
28	734.906	138.149	7.3401	218.868	58.429	Case-2
30	736.928	138.861	7.3693	220.097	58.502	Case-2
35	1167.29	381.729	11.6729	691.971	81.273	Case-3
45	1263.11	534.183	12.6311	997.171	86.627	Case-3

system for a preventive maintenance cycle for no just-in-time buffer. The expected unit total cost for no just-in-time buffer is always higher than that with buffer inventory. It increases our strength of belief that proper just-in-time structure always requires some inventory for its efficient operation. The column (Q_1/k_1) indicates the required time for buffer replenishment. It is found that as the demand of imperfect quality items increases, the buffer replenishment time and the buffer stock increase. This is quite obvious because for higher demand during preventive maintenance larger amount of imperfect quality items is required. $E\pi^*(Q_1)$ for $D_2 = 20$, 25, 28, 30 and 35, 45 are obtained by solving Eqs. (8)–(10), respectively, depending on the relations between D_2 and expected amount of imperfect quality items. It is found that $E\pi^*(Q_1)|_{D_2=35} < E\pi^*(Q_1)|_{D_2=45}$ because higher demand of imperfect quality items introduces higher amount of shortage cost for imperfect quality items for the entire preventive maintenance cycle. However, the expected unit system cost for $D_2 = 20$ is higher than that for $D_2 = 25$. Because, $\left|E\left(\frac{pD_1}{1-p}\right) - D_2\right|_{D_2=20} > \left|E\left(\frac{pD_1}{1-p}\right) - D_2\right|_{D_2=25}$. Thus the average amount of inventory for $D_2 = 20$ is higher than that of $D_2 = 25$, which introduces more holding cost for imperfect quality items and hence higher expected system cost occurs. Here, optimal buffer inventory and corresponding expected unit system cost are determined under variable buffer replenishment rate, where it is assumed that the maximum production capacity $K = 1000$. It is found, as we predicted in the previous section, that if $D_2 < E\left(\frac{pD_1}{1-p}\right)$ then the buffer replenishment rate will attain its maximum capacity to reduce the holding cost corresponding to the average amount of buffer inventory. It is also found from Table 2 that expected unit system cost for a particular D_2 is considerably low with variable buffer replenishment rate than that with fixed buffer replenishment rate. The buffer replenishment time with variable buffer replenishment rate increases as the demand of imperfect quality item increases. In Table 2 the optimal buffer replenishment rate beyond its capacity is determined depending on the demand of imperfect quality items. Thus in the decision making situation if a system produces imperfect quality items in random proportion then its always preferable to adjust the buffer replenishment rate beyond its capacity instead of considering fixed buffer replenishment rate. In Tables 3 and 4, some sensitivity of the model is performed by considering fixed and variable buffer replenishment rate. The parameter values are changed −40%, −20%, 20% and 40%, respectively, once at a time keeping unchanged the remaining parameters. It is found that the model is moderately sensitive for the change in the parameter values T, h_1, S_p and S_i and lowly sensitive for the error in the estimation of the parameter value h_2. Thus proper attention is needed for the estimation of the values of T, h_1, S_p and S_i in the decision making context.

5. Summary and concluding remarks

In the existing literature, plenty of maintenance policies and inspection policies are available to cope with the production of imperfect quality items for imperfect repair during preventive maintenance and for the production of imperfect quality items due to the out-of-control state of the production unit. But, in this paper, we developed a model under the basic assumption that the system produces imperfect quality items or produces items not to the standard maintained by the production unit in random proportion due to some uncontrollable factors of production (frequently found to happen in polymer, cotton and leather industries) and completely independent of the nature of the preventive maintenance under the just-in-time configuration.

Table 2 Optimal values of Q_1, k_1 and $E\pi$ for variable buffer replenishment rate.

D_2	Q_1^*	k_1^*	$E\pi(Q_1^*)$	$Q_1 = k_1$	% change in $E\pi^*$ for variable replenishment rate	Remarks
20	1158.44	500	214.858	2.3169	−33.043	Case-1
25	1222.36	500	151.44	2.4447	−47.793	Case-1
28	1175.1	229.842	110.443	5.1126	−20.465	Case-2
30	1152.38	213.102	113.237	5.4076	−18.4529	Case-2
35	1101.16	180.964	119.478	6.0849	−82.7337	Case-2
45	1381.51	339.471	171.055	4.0659	−82.846	Case-2

Table 3 Sensitivity analysis for fixed buffer replenishment rate (for $D_2 = 25$, $k_1 = 100$).

Parameters	% change in parameter value	% change in Q_1^*	$E\pi^*(Q_1^*)$	% change in $E\pi^*(Q_1^*)$
T	−40	3.226	235.076	19.062
	−20	1.695	208.221	5.46
	20	−1.781	195.257	−1.1056
	40	−3.612	198.211	0.39
h_1	−40	29.579	175.676	−11.023
	−20	12.86	187.96	−4.8015
	20	−10.208	204.982	3.8199
	40	−18.5117	211.129	6.9332
h_2	−40	0.8049	188.518	−4.5188
	−20	0.4019	192.98	−2.2589
	20	−0.4009	201.897	2.2574
	40	−0.8008	206.352	4.5138
S_p	−40	−31.8491	158.909	−19.5153
	−20	−14.4964	179.881	−8.8933
	20	12.3036	212.366	7.5598
	40	22.8848	225.216	14.0681
S_i	−40	3.0656	176.38	−10.6665
	−20	1.5298	186.927	−5.3247
	20	−1.5243	207.918	5.3069
	40	−3.0426	218.362	10.5966

Table 4 Sensitivity analysis for variable buffer replenishment rate (for $D_2 = 35$).

Parameters	% change in parameter value	% change in Q_1^*	% change in K_1^*	$E\pi^*(Q_1^*)$	% change in $E\pi^*(Q_1^*)$
T	−40	17.1674	87.9307	150.448	25.9211
	−20	8.0177	31.3598	132.709	11.074
	20	−5.2781	−15.4959	109.101	−8.6853
	40	−4.2328	−15.4959	102.266	−14.406
h_1	−40	21.748	−2.8276	91.6392	−23.3004
	−20	10.0076	−0.5023	106.993	−10.4998
	20	−8.6808	−0.6482	129.85	8.6811
	40	−16.3034	−2.0479	138.598	16.0029
h_2	−40	0.0327	0.0271	119.45	−0.0234
	−20	0.0163	0.0138	119.464	−0.0117
	20	−0.0154	−0.0133	119.492	0.0117
	40	−0.0318	−0.0271	119.506	0.0234
S_p	−40	−27.4573	−15.4959	96.4152	−19.303
	−20	−14.6076	−12.9241	109.679	−8.2015
	20	10.8622	9.6815	126.661	5.9701
	40	19.2161	17.1857	132.141	10.5986
S_i	−40	7.9498	33.2287	108.676	−9.041
	−20	3.5099	13.1065	114.738	−3.9673
	20	−2.9033	−9.3875	123.363	3.2516
	40	−4.8867	−15.4959	126.656	6.0078

Quite often, it is observed in polymer, cotton and leather industries that a percent of the finished goods are defective due to imperfect operations at any stage of the production processes. The items are marked imperfect, after screening process, in respect of quality factors for branding such as quality of materials, colors, accurate shape and size, sewing, etc. These imperfect, i.e., defective items are purchased at lower price compared to the perfect quality items at the secondary shops/enterprises.

It is also assumed that, after preventive maintenance, the quality of the products remains same as before. The optimal just-in-time buffer for perfect and hence imperfect quality items are determined by considering fixed buffer replenishment rate and variable buffer replenishment rate beyond its capacity to minimize expected unit system running cost. A numerical example is presented which illustrates that, in just-in-time structure; some buffer inventory must be needed to minimize the system running cost. The demand of imperfect quality items has a significant effect on the expected system running cost. It is also found that the adjustment of buffer replenishment rate beyond its capacity depending on the expected amount of imperfect quality items leads to lower system running cost than the predetermined fixed buffer replenishment rate. In the decision making context, for lower demand of imperfect quality items in comparison to the demand of perfect quality items, it is always better to adjust the buffer replenishment rate. However, it is assumed that T is fixed. An interesting area of further investigation is to determine T, k_1 and Q_1 simultaneously which would be generalization of the model presented here. This also determines the joint effects of unavoidable random amount of imperfect quality items and the imperfect quality items produced due to the out-of-control state of the production unit on the expected unit system running cost. The paper provides an elementary idea about how production of perfect quality product is affected by imperfect quality product which is produced due to error in the production unit. In the current business era, imperfect quality product has a good demand. Thus, further investigation is essential to obtain maximum benefit from this market through proper inventory policies for imperfect quality product.

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