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Qualitative Voting*

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Resumen

¿Podemos encontrar un mecanismo sin transferencias monetarias que permita a los votantes expresar la intensidad de sus preferencias? ¿Puede la opinión de una minoría ser respetada cuando la intensidad de sus preferencias es muy superior a la intensidad de las preferencias de la mayoría? En este artículo proponemos una alternativa al sistema común de votación (una persona - una decisión - un voto) en la que cada agente dispone de un cierto número de votos que pueden ser libremente distribuidos entre una número predeterminado de decisiones. Lo novedoso de este sistema de votación es que permite a los votantes expresar cuánto les importa cada decisión. Caracterizamos condiciones suficientes en las que este sistema de votación es óptimo y superior a la regla de la mayoría.

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Códigos JEL: C72, D70, P16

Abstract

Can we devise mechanisms that allow voters to express the intensity of their preferences when monetary transfers are forbidden? Can minorities be decisive over those issues they feel very strongly about? As opposed to the usual voting system (one person-one decision-one vote), we propose a voting system where each agent is endowed with a fixed number of votes that can be distributed freely among a set of issues that need to be approved or dismissed. Its novelty relies on allowing voters to express the intensity of their preferences in a simple manner. This voting system is optimal in a well-defined sense: in a strategic setting with two voters, two issues and preference intensities uniformly and independently distributed across possible values, Qualitative Voting Pareto dominates Majority Rule and, moreover, achieves the only ex-ante optimal (incentive compatible) allocation. The result also holds true with three voters as long as the voters preferences towards the issues differ sufficiently.

Keywords: Voting, Intensity Problem, Alternatives to Majority Rule, Conflict Resolution

JEL Classification: C72, D70, P16

> The history of economic institutions shows a great deal of change, facilitating economic activities that would have earlier been impossible. No similar development and change has occurred in the political system; yet the need for such facilitation is undoubtedly equally great
>
> JAMES COLEMAN (1970)

1. INTRODUCTION

Voting is the paradigm of democracy. It reflects the desire to take everyone's opinion into account rather than imposing, by whatever means, the decision of a particular individual. At its root lies the belief that people should be allowed to cast their votes freely and, above all, that they should be treated equally[1]. In stark contrast to many economics situations, side-payments are therefore ruled out in order to maintain ex ante equality of voting power despite an unequal distribution of material wealth.

Although a variety of voting rules have been proposed to suit different settings, in practice Majority Rule (MR, hereafter) is used almost exclusively. From an economist's perspective, and given that most of our work is built on the diverse behaviour of individuals with different marginal propensities to consume, produce, etc., the main difficulty with MR is that it does not capture the intensity of voters' preferences. Just as we contemplate the importance of the *willingness to pay* in the provision of public goods, we would expect that taking into account the *willingness to influence* in a voting situation will increase efficiency.

Previously, responses to this criticism have argued that if we were to treat a very enthusiastic voter and a very apathetic one differently, equality would no longer hold.[2] However this reasoning is too narrow. In this paper we show that we can build a very simple voting rule that allows voters to express intensity and reach in some situations a strictly Pareto superior allocation than the one achieved by MR while preserving equality among voters.

Following Coleman's quote, we wish to stimulate the debate over the developments that should occur in our political institutions to make them better able to represent and govern our societies. We want to consider voting systems where the concept of decisions *preferred by most members* is replaced by that of decisions *most preferred*

[1] See, for instance, Locke (1690).

[2] See Spitz (1984).

by members; we want votes to have an embedded quality which is somehow linked to the intensity of the voters' preferences; ultimately, we want to show the circumstances under which the strategic interactions between voters do not undermine the gains from expressing their willingness to influence.

In a setting with a closed agenda of N issues that have to be approved or dismissed, we propose a *Qualitative Voting* rule (QV, hereafter) that allows voters to simultaneously and freely distribute a given number of votes among the issues. In this way we are providing voters with a broader set of strategies than the classical «one person — one decision — one vote» but at the same time preserving equality since all individuals are endowed with the same ex-ante voting power. We must stress that the Qualitative Voting rule resembles other mechanisms (see the Related Literature Section below and Chapter 8 in Mueller (2003)); yet, its analysis in a fully strategic setup and its application to a situation with binary decisions is novel to this paper.

Essentially, QV introduces two main improvements on the usual voting rules. On the one hand, it answers the classical debate in the political science literature on «the problem of intensity» by allowing strong minorities to decide over weak majorities. Secondly, it allows voters to trade off their voting power, adding more weight to the issues they most care about, and so unlocks conflict resolution situations.

The latter intuition is best captured by the following situation: imagine two voters with opposing views in two issues but such that the first (second) voter mostly cares about the outcome on the first (second) issue. QV allows each of them to decide on their most preferred issue and hence non-cooperatively coordinate on a Pareto optimal allocation. We can devise many different instances in which such situations occur and where side payments may not be possible (or may be forbidden): an international dispute, a bilateral agreement in arms/pollution reduction, a country having the two legislative chambers governed by opposing parties,[3] a clash between the management

[3] The US Congress and Senate have repeatedly been in a situation with a Republican majority in one chamber, and a Democratic majority in the other, consequently many bills have been vetoed by one chamber and decision-making has been difficult. QV could have made the process more efficient by allowing each party to support those bills which its electorate felt more strongly about. Money and Tsebelis (1997) claim that the gains we expect from the use of QV may already be observed through the existence of committees: «One essential assumption of distributive theories of Congress is that the policy space is multidimensional. This is how committee chairs and members extract gains from trade. They give up their positions in the less important dimension in order to gain in the more important one, their own jurisdiction.»

and the union of a particular firm, etc.[4] A rigorous analysis of such voting system is essential in light of the fact that a variant of it is currently being used to settle disputes (see the article «March of the robolawyers» in The Economist, 9/3/2006).

The goal of the present paper is not only to compare QV to MR but, also to assess its optimality. We therefore take a mechanism design approach that allows us to characterise the optimal allocations from amongst the implementable ones. This requires us to look at highly stylised scenarios with preferences over issues being discrete. The discrete setting implies, in turn, that we cannot look at very general scenarios. We first look at a conflict resolution situation with two voters, two issues and independent uniform priors on the voters' preferences, and see that non-indifferent voters allocate all their votes on their most preferred issue. Moreover, we prove in Theorem 1 that QV reaches the only incentive compatible ex-ante optimal allocation.

We then extend our setting to analyse the potential use of QV in committees —and its role in allowing minorities to be decisive— by extending the previous setting and considering three voters instead of two. In equilibrium, voters shift their voting power towards the issue they value most, and QV is optimal when voters value their most preferred issue at least three times as strongly as their least preferred one (Theorem 2).[5]

Examples follow the statement of both theorems in order to illustrate the results and shed some light into the applicability of QV into the real world. The dependence of the results on the independent uniform priors is proved to be critical in Section 8: the more deterministic the priors are, the more strategically voters react and, consequently, the more difficult it is to achieve a truthful revelation of preferences. Moreover, the strategic interactions between individuals may lead to the non-existence of pure-strategy equilibria in the game induced by QV. However, this does not undermine the results of this paper: there are some situations in which one can strictly Pareto-improve on the allocation achieved by MR through a simple mechanism we have called QV.

[4] Our setting can be reinterpreted as a non-zero sum Colonel Blotto Game (two colonels are fighting over some regions and have to decide how to divide their forces; the one with larger forces wins the region and the winner of the battle is the one with the most won territory). If the colonel is not indifferent between winning two different regions, the payoff of the game is not only contingent on how many regions he has won or lost but precisely on which regions he has won or lost. Myerson (1993) also refers to the Colonel Blotto Game when analysing the incentives for candidates to create inequalities among voters by making heterogeneous campaign promises.

[5] There is also an equilibrium that replicates the majoritarian outcome. Nevertheless, this equilibrium does not survive any usual refinement.

In the remainder of this section we review the existing literature and relate our model to this earlier work. The rest of the paper is organised as follows: Section 2 introduces the model, Section 3 analyses the allocation achieved by QV in a situation with two voters, Section 4 extends the analysis to a situation with three voters, Section 5 analyses the optimality of QV, Section 6 presents two examples, Section 7 briefly discusses some of the results, and Section 8 concludes.

1.1. Related literature

Intensity of preferences can play a role in voting games only if we move away from unidimensional settings and allow voters to trade-off their voting power across issues. The gains we expect from the use of rules such as QV come precisely from non-homogeneous preferences across issues.[6] Accordingly, our work belongs to a wider set of models with two key features: heterogeneous preferences and a multidimensional setting. The case for QV under these conditions rests on a simple comparative advantage argument: in the same way as each country should specialise by focusing on the sectors of the economy where it is relatively more productive, QV allows voters to decide on the issue they care relatively more about. However, this leaves open the important question of implementation.

The two papers most closely related to ours are Jackson and Sonnenschein (2007) and Casella (2005). Jackson and Sonnenschein (2007) show that linking decisions normally leads to Pareto improvements. More specifically, they present a simple rule that achieves the ex-ante efficient allocation and induces truthful revelation as we increase the number of decisions. Such rule is very simple in the sense that it just requires voters to match their voting profiles to the frequency of preferences induced by their prior distribution. The key differences with our work is that they provide an efficiency result in the limit for a particular indirect mechanism and their action space depends on the prior distribution of preferences. Instead, we provide an indirect mechanism which

[6] Bowen (1943) has already pointed out that MR is an efficient mechanism whenever the voters' intensity of preferences are disbributed symmetrically. Similarly, Philipson and Snyder (1996) analyse an organised vote market and show that its efficiency gains (relative to MR) are larger the more heterogeneous the preferences are.

does not depend on the prior distribution, and characterise its optimality in a particular setting.[7]

Casella (2005) proposes a system of Storable Votes to be used in situations where voters have to decide over the same binary decision repeatedly over time and shows its superiority with respect to MR in a particular setting. Our framework is different in the sense that voters simultaneously cast all their votes and know their full preference profile at the time of voting (no time dimension). Moreover, we undertake a mechanism design analysis which allows us not only to compare two particular voting rules but also to characterise all implementable allocations and, from them, identify the optimal ones.

Most of the literature on mechanism design without transfers (and most of the literature on voting) is built in a setting with ordinal utilities where one alternative has to be selected out of many, i.e. a setting of electing representatives.[8] Within that literature, QV has the flavour of a scoring rule (specially cumulative voting) though there is a crucial distinction:[9] a scoring rule is used to elect one representative out of many, instead QV deals with a series of binary elections where N independent issues have to be approved or dismissed.

The literature on alternatives to MR is related to our work insofar as it provides mechanisms which capture the intensity of the voters preferences but their complexity undermines its applicability (see the chapter entitled «Complicated Alternatives to Majority Rule» in Mueller (2003)). On the one hand, Tideman and Tullock (1976) develop an application of the Clarke-Groves mechanism to a voting framework. Needless to say, this requires monetary transfers and hence fails to satisfy the equality property. On the other hand, Hylland and Zeckhauser (1979) propose a Point Voting Rule to be used for the contribution to public goods, with perfectly divisible points.[10] They focus

[7] Assessing trade-offs between issues and extracting all possible gains from differences is also one of the main concerns of the negotiation analysis and the international relations literatures. See for instance Keeney and Raiffa (1991). Closer in spirit to our work, Shepsle and Weingast (1994, pg 156) assert that «The political solution is to create an institutional arrangement for exchanging support that is superior to a spot market». Likewise, Levy (2004) models political parties as being able to exploit the gains from differing relative valuations in a multidimensional policy space.

[8] See Gibbard (1973) and Satterthwaite (1975).

[9] «In a scoring rule, each voter's ballot is a vector that specifies some number of points that this voter is giving to each of the candidates (or parties) that are competing in the election. These vote-vectors are summed over all voters, to determine who wins the election», Myerson (1999), pg. 673-674.

[10] Brams and Taylor (1996) propose a *Point Voting Rule* (the *Adjusted Winner Procedure)* that is essentially our voting system in a setting of a conflict resolution. Their weakness, though, is that

on providing an (arbitrary) social choice function that induces the truthful revelation of preferences.

When we imagine a way in which politicians give more weight to a particular position we immediately think of logrolling or vote trading. The relationship between these practices and QV, and the advantages of the latter as a way of expressing willingness to influence, are briefly discussed in Section 7.

2. The model

A voting game is defined as a situation where I voters have to dismiss or approve N issues and no monetary transfers are allowed. Voters privately know their preference profile across the N issues and the prior distributions from which these preferences are drawn are common knowledge. From a mechanism design perspective this is a private values multidimensional problem with multilateral asymmetric information and no transfers.

Voters and issues are denoted i∈{1,2,...,I} and n∈{1,2,...N}, respectively. The value voter i places on issue n is θ_n^i. The preference vector of voter i is $\theta_n^i = (\theta_1^i, \ldots, \theta_N^i \in \Theta \subseteq \Re^N$, i = $\forall 1, \ldots, I$.

Preferences should be interpreted as follows: a positive type ($\theta_n^i > 0$) wants the issue to be approved, a negative one ($\theta_n^i < 0$) wants it to be dismissed and the absolute value ($|\theta_n^i|$) captures the intensity of the preference towards that particular issue.

Voter i's payoff from the decision on a particular issue, n, is as follows: θ_n^i if the issue is approved; and, ($-\theta_n^i$) if the issue is dismissed. The total payoff is the sum of the individual payoffs across the N decisions.[11]

An allocation is a vector of the probabilities that each of the N issues is approved. The set of allocations is defined as $X = \{(p_1, \ldots, p_N): p_1, \ldots, p_{N_i} \in [0,1]\}$ where p_n is the probability issue n is approved; a voter with preferences θ^i obtains utility $u(p, \theta^i)$ from allocation $p_N \in X$.

$$u(p,\theta^i) := \sum_{n=1}^{N} p_n \theta_n^i + (1 - p_n)(-\theta_n^i) = \sum_{n=1}^{N} (2p_n - 1)\theta_n^i$$

they do not take into account the strategic interactions by assuming players are honest.

[11] The definition of payoffs implicitly assumes that issues are independently valued. That is, there are no complementarities between them. Provided that issues are independently valued, results can be extended to any linear transformation of the payoffs.

We are in a setting of private values where each agent's utility depends only on his own type and utilities are multilinear. Voters cast their vote at the interim stage; that is, when they know their own preferences but only know the prior distribution from which their opponents' preferences are drawn. The strategy space defined by QV consists of mappings from the set of individual preference profiles to the set voting profiles V

$$V: = \{(s_n, v_n)_{n=1,\dots,N} : s_n \in \{-1,1\} \text{ and } v_n \in \{0,1,\dots,V\} \text{ and } \textstyle\sum_n v_n = V$$

where each voting profile specifies for each issue whether the voter wants it to be approved ($s_n = 1$) or dismissed ($s_n = -1$) as well as the number of votes (v_n) allocated to it.

QV aggregates the voting profiles in the following way: whenever the total number of votes in favour of the approval (dismissal) of an issue is larger than the number of votes in favour of the dismissal (approval), the issue is approved (dismissed). Ties are resolved applying the usual MR. The relevance of the tie breaking rule is explained in Section 4.1. Briefly,

$$\begin{cases} \sum_i s^i_n v^i_n > 0 \Rightarrow \text{issue n is approved} \\ \sum_i s^i_n v^i_n < 0 \Rightarrow \text{issue n is approved} \\ \sum_i s^i_n v^i_n < 0 \Rightarrow \text{MR is applied} \end{cases}$$

for every $n = 1,\dots,N$.

We want to characterise the Bayesian Nash equilibria of the game induced by QV. In order to assess the optimality of QV we need to characterise the set of implementable mechanisms and select the ones that maximise the sum of utilities.[12] In order to do this, we simplify our setting and throughout consider a setting with (i) two or three voters (I = 2 or 3), (ii) two issues (N=2), (iii) four valuations per issue ($\theta^i_n \in \{-1, -\theta, +\theta, +1\}, \theta \in (0,1)$)[13] and (iv) uniform and pairwise independent priors: Pr $\{\theta^1_n = 1\}$ =

[12] In Hortala-Vallve (2009) we have proved that any implementable mechanism is characterised by inducing indirect utilities that are homogeneous of degree one and convex. We have not been able to solve the optimisation problem while imposing the homogeneity condition.

[13] Note that without loss of generality and in order to simplify the notation we have assumed the high issue to take a value equal to one. The analysis is totally analogous to the more general setting where

$$\theta^i_n \in \{\pm\bar{\theta}, \pm\underline{\theta}\}\ \bar{\theta} > \underline{\theta} > 0.$$

Pr $\{\theta_n^1 = -1\}$ = Pr $\{\theta_n^1 = \theta\}$ = Pr $\{\theta_n^1 = -\theta\}$ = 1/4 and there is pairwise independence across issues and voters.

We define the set of a voter's preference profiles as $\Theta := \{-1, -\theta, +\theta, +1\} \times \{-1, -\theta, +\theta, +1\}$.

3. On the Use of QV in Conflict Resolution Situations

The case with 2 voters introduces the main effect of QV as voting rule. It allows voters to trade-off their voting power. Specifically, when we consider two issues it allows voters to rank the issues and reach the only ex-ante optimal allocation. The next example best captures this intuition:

> Example: Two friends, Anna (i = 1) and John (i = 2), are spending an evening together. Above all they want to be together, however, they can't come to an agreement. Anna wants to see a horror film and would like to have dinner in a new Italian restaurant while John prefers a comedy film and eating sushi in a Japanese restaurant (i.e. we have two linked battle of the sexes games) Following the previous notation, issue 1 corresponds to the film decision (p_1 is the probability of seeing the horror film and $1 - p_1$ the probability of seeing the comedy one) and issue 2 to the restaurant decision (p_2 is the probability of the Italian restaurant and $1 - p_2$ the probability of the Japanese one). If they vote on each of the issues nothing is decided and they have to stay at home (which we assume is not optimal for either of them). Additionally, suppose that Anna really cares about the restaurant decision while John cares more about the film (i.e. $\theta^1 = (\theta,1)$ and $\theta^2 = (-1,-\theta)$). It seems sensible that, as good friends, each of them will give up on their least preferred option; they will both go to the Italian restaurant and the comedy film yielding an overall utility of $2(1 - \theta) > 0$. From a game theoretic perspective, they are both coordinating on the Pareto optimal allocation that maximises the sum of utilities. QV is precisely a mechanism that allows voters to coordinate non-cooperatively on the only ex-ante optimal outcome.

Voters are endowed with V>0 votes that can be freely distributed between the two issues. We assume that V is even so that voters can split the votes evenly between the two issues if necessary. The uniform and independent priors on the opponent's preferences imply that it is an optimal strategy for each player to declare the true sign of his preferences, i.e. $s^i_k = sign\ (\theta_k^i)$.

Without loss of generality, we analyse the optimal strategy of voter i whenever he has positive preferences and he casts v votes in issue 1. His payoff is:

$$\left(\frac{1}{2}+\frac{1}{2}\tilde{p}_1 (v \mid \theta_1^j > 0)\theta_1^i\right) + \left(\frac{1}{2}+\frac{1}{2}\tilde{p}_2 (V - v \mid \theta_2^j > 0)\theta_2^i\right)$$

The previous expression captures the property that unanimous preferences are implemented and $P_n\ (v)$ is the expected value of $(2p_n - 1)$ whenever voter i casts v votes. They are defined as follows (conditional probabilities are omitted for notational simplicity):

$$\tilde{P}_1(v) := 2\left(\Pr\ (v - v_1^j > 0) + \frac{1}{2}\Pr\ (v - v_1^j = 0)\right) - 1$$

$$\tilde{P}_2(V - v) := 2\left(\Pr\ (V - v - (V - v_1^j\ ) > 0 + \frac{1}{2}\Pr\ (V - v - (V - v_1^j\right) = 0 - 1$$

Restricting our attention to symmetric strategies implies that we can rewrite the payoff of voter i as

$$\theta_2^i + (\Pr\ (v + v_1^j > 0) + \Pr\ (v + v_1^j = 0)\ (\theta_1^i - \theta_2^i)$$

Voter i wants to maximise the expression inside the curly brackets whenever $\theta_1^i > \theta_2^i$, i.e. $v_1^i = V$.[14] He wants to minimise it when $\theta_1^i < \theta_2^i$, i.e. $v_1^i = 0$. The previous expression shows that a voter that equally weights both issues will be indifferent among any voting profile. In what follows we will call a player that values both issues with equal intensity, $\theta_1^i = \theta_2^i$, an *indifferent* player.

Summing up, the equilibrium strategy for a player with positive preferences has $s^i = (1,1)$ and

$v^i\ \ (V, 0)$ when $\theta_1^i > \theta_2^i$

[14] Player i want to invest a number of votes strictly higher (if possible) than the absolute value of his opponent's invested votes on the first issue. Taking into account that player j plays accordingly; the only equilibrium has non-indifferent players investing all their voting power on their preferred issue.

$v^i \ (\frac{V}{2}, \frac{V}{2})$ when $\theta_1^{\ i} = \theta_2^{\ i}$

$v^i \ (0, V)$ when $\theta_1^{\ i} < \theta_2^{\ i}$

Hence, QV allows voters to sometimes be decisive in the issue they value most.

4. On the Use of QV in Committees

We depart now from a pure conflict resolution situation and consider a setting with three voters. In the previous analysis each voter was seeking to counteract the votes invested by his opponent. This effect is still in place but now we have an additional element: in some situations some voters may not be pivotal. Similarly, with two voters the tie breaking rule had no welfare effects. However, with three voters the tie breaking rule plays a crucial role and has important welfare effects.

Voters are endowed with an even number of votes V. With uniform and independent priors it is still an optimal strategy for each voter to truthfully declare his preference between the approval and dismissal for each issue. We will now focus on symmetric pure strategy equilibria —i.e. all voters play the same strategy.

We want to focus on the set of final allocations reached in equilibrium rather than the set of equilibria. For this purpose we introduce the term essential as an equivalence class of equilibria that induce the same allocation —notice that given the nature of our game there are situations where some votes are not pivotal and hence can be placed anywhere without affecting the outcome.

The following Lemma states that the strategy followed by any voter is independent of the labelling of the issues. That is, the strategy of a non-indifferent voter is summarised by a parameter $\gamma \in \{0,1,...,V\}$ which should be interpreted as the number of votes invested in his most preferred issue; $(V - \gamma)$ are the votes invested in his least preferred issue. The Lemma also states that indifferent voters should divide their votes equally in a symmetric equilibrium.

Lemma 1: *In a setting with two issues, three voters and uniform and independent priors, any symmetric pure strategy essential equilibrium satisfies the following two properties:*
1- Non-indifferent voters always invest $\gamma^* \in \{[V/2] ,..., V\}$ *in their most preferred issue.*
2- Indifferent voters split their votes evenly. That is, they invest (V/2) votes on each issue

Note that the equivalence class allows for different strategies but they all achieve the same outcome. The proof (which is provided in the appendix) relies on showing that an equilibrium which relies on the labelling of the issues cannot be sustained. Imagine, for instance, that there exists an equilibrium where indifferent voters cast more votes on the first issue: $\gamma^*_{ind} > V/2$. Any voter would now be better off by deviating and casting $V - \gamma^*_{ind}$ on the first issue —the fact that more votes are cast in issue 1 reduces the probability that a single vote will be pivotal thus it is optimal to shift votes towards issue 2.

Proposition 1 *In a setting with two issues and three voters, there are essentially three symmetric pure strategy equilibria. These are:*

	$\gamma^* = V$	$\gamma^*_{ind} = V/2$	–all votes on preferred issue
	$\gamma^* = V/2$	$\gamma^*_{ind} = V/2$	–equivalent to MR
when $\theta = 1/2$	$\gamma^* = V$	$\gamma^*_{ind} = V/2$	–all votes on preferred issue

where γ^* *is the number of votes invested by non-indifferent voters in the most preferred issue and* γ^*_{ind} *is the number of votes invested by indifferent voters in issue one.*

The proof of the Proposition is quite tedious and is left to the appendix. Its difficulties lie in the essential aspect of it; we can devise many possible combinations of votes where no individual is better off by deviating but where some votes are not pivotal and hence can be placed in any of the issues. The first equilibrium allows strong minorities to impose their will over weak majorities. The second equilibrium replicates the MR allocation. For future reference they will be called Equilibrium QV (EqQV) and Equilibrium MR (EqMR), respectively. Finally, the third equilibrium can be seen as a mid point between the other two where a weak majority with an indifferent voter can overcome a strong minority. This equilibrium only holds for a particular value of θ and does not survive any robustness check.[15]

[15] This equilibrium disappears whenever we consider the continuous valuation of the issues (see Section 7). There are two reasons for this: (1) the relative intensity for which it holds has measure zero in the continuous case (given uniform preferences) and (2) the strategy followed by indifferent players is crucial for this equilibrium to hold and these voters have in general zero measure in the continuous case.

Since Proposition 1 holds for any number of votes, it will hold when votes are perfectly divisible.[16] The multiplicity of equilibria can be resolved by noticing that EqQV is the one that survives any usual refinement: if voters randomly allocate some of their votes, non-indifferent voters will seek to reduce their probability of losing (i.e. increase their probability of being pivotal) their most preferred issue by investing all their votes in it. Henceforth we focus our attention on EqQV.[17]

4.1. The Tie Breaking Rule

The tie breaking rule plays a crucial role in the three voters' case and has important welfare effects. Consider the probability of a favourable outcome under MR. Given the uniform priors assumption, a voter observes his will being implemented on any issue with probability (3/4) since the issue can only be dismissed if the remaining two voters are opposed to him —an event with probability (1/4). Imagine now, that the tie breaking rule under QV is the toss of a fair coin, i.e. the issue is approved with probability (1/2): the probability of favourable outcome decreases, (1/2)<(3/4). The optimal tie breaking rule should maximise the probability of a favourable outcome for any player in an incentive compatible way.

We show that in case of ties, issues should be decided through the usual MR. QV becomes a voting rule that allows issues to be decided on the grounds of the total intensity of preferences. In case the intensity of preferences is not decisive, the issue is approved on the basis of overall support (MR). QV happens to be a natural extension of the usual voting rule where voters declare their position with respect to the approval or dismissal of an issue and then invest extra votes to reflect their *willingness to influence*.

[16] In Section 7 below, we show that in the case with continuous valuation of issues and perfectly divisible votes, the EqQV and EqMR are the only equilibria.

[17] The multiplicity of equilibria when analysing different mechanisms is usually resolved by selecting the best equilibrium in each possible situation. Note that this approach would benefit our analysis because MR would never be able to do better than QV given that the latter can also achieve the allocation reached by the former. Therefore, focusing on the first equilibrium makes our optimality analysis more difficult.

5. Optimality Analysis

In order to characterise the optimality properties of QV we first need to characterise the set of implementable mechanisms. The Revelation Principle allows us, without any loss of generality, to restrict the analysis to the study of direct revelation mechanisms. A direct revelation mechanism is a mechanism where the set of messages which players may send coincides with the space of preference parameters. The mechanism maps these revelations into an allocation ($p : \Theta^I \to X$).

We focus our attention on the set of mechanisms that preserve unanimous wills, have no systematic tendency towards the approval or dismissal of any of the issues (neutrality) and treat all individuals in the same manner (anonymity). Moreover, given that we are in a multidimensional setting we want all the issues to be treated equally too. It will be useful to define a mechanism as being reasonable whenever it satisfies the previous four properties (notice that MR satisfies them).

Definition 1 *A mechanism* $p : \Theta^I \to X$ *is* reasonable *if and only if it satisfies*

1. *Unanimity:* $p_n(\theta^1,\ldots,\theta^I) = \begin{cases} 1 \text{ if } \theta^i_n > 0, \forall i \\ 0 \text{ if } \theta^i_n < 0, \forall i \end{cases}$ *for any n.*

2. *Anonymity:* $p_n(\theta^1,\ldots,\theta^I) = p_n(\theta^{\sigma(1)},\ldots,\theta^{\sigma(I)})$ for any n and $\sigma \in s_i$.

3. *Neutrality:* $p_n(\theta^1,\ldots,\theta^I) = 1 - p_n(-\theta^1,\ldots,-\theta^I)$ for any n.

4. *Neutrality across issues:* $\forall n=1,...,N$ *and* $\xi_m \in \{+1,-1\}$, $\forall m=1,...,N$
   $p_n(\theta^1,\ldots,\theta^I) = p_n((\xi_1\theta_1{}^1,\ldots,\theta_n{}^1,\ldots,\xi_N\theta_N{}^1),\ldots,(\xi_1\theta_1{}^I,\ldots,\theta_n{}^I,\ldots,\xi_N\theta_N{}^I))$.

5. *Symmetry across issues:* $\forall n=1,...,N, \forall\sigma \in S_N$,
   $p_n(\theta^1,\ldots,\theta^I) = p_{\sigma(n)}((\theta^1{}_{\sigma(1)},\ldots,\theta^1{}_{\sigma(N)}),\ldots,(\theta^I{}_{\sigma(1)},\ldots,\theta^I{}_{\sigma(N)}))$.

where S_k *denotes the set of all possible permutations of k elements.*

5.1. Implementable mechanisms

We want to characterise all mechanisms that induce a truthful Bayesian Nash equilibrium at the interim stage —the moment where each agent knows his own type (but only holds beliefs on his opponents' types) and needs to reveal his type in the direct mechanism or cast his votes in the indirect mechanism. The interim utility of a voter that declares θ^i while his type is θ^i ,is defined as:

$$u\,(\hat{\theta}^i, \theta^i) := E_{\theta^{-i}}\,\{u(p(\hat{\theta}^i, \theta^i)\;\theta^i)\}$$

where, $\theta^{-i} := (\theta^i,...,\theta^{i-1},\theta^{i+1},...,\theta^I)$. Note that this is simply his expected utility under the assumption that his opponents will truthfully reveal their types. To simplify the notation let us also define the interim prospect on issue n as $p_n\,(\hat{\theta}^i) = E_{\theta^{-i}}\,\{2p_n(\hat{\theta}^i, \theta^{-i}) - 1\}$.[18] The interim utility now reads as $u(\cdot, \theta) = P_1(\cdot)\theta_1 + P_2(\cdot)\theta_2$.

Implementable mechanisms are the ones that satisfy Incentive Compatibility constraints (IC) —it is optimal for each type of voter to reveal his true type. Restricting the analysis to the set of reasonable mechanisms, together with the uniform and independent priors implies that we just need to analyse the ICs from the perspective of a voter that positively values both issues. Condition 5 for a mechanism to be reasonable also implies that $P_2\,(\theta_1{}^i,\theta_2{}^i) = P_1\,(\theta_2{}^i,\theta_1{}^i)$. The three possible interim utilities can now be expressed in terms of P_1:

A non-indifferent type:	$P_1\,(1,\theta)\cdot 1 + P_1\,(\theta,1)\cdot\theta$
An indifferent high type:	$P_1\,(1,1)\cdot 1 + P_1\,(1,1)\cdot 1$
An indifferent low type:	$P_1\,(\theta,\theta)\cdot\theta + P_1\,(\theta,\theta)\cdot\theta$

The next Proposition tells us the conditions that any reasonable mechanism should satisfy in order to be implementable (note that the Proposition holds for an arbitrary number of voters).

Proposition 2 *A reasonable mechanism* $p : \Theta^I \to X$ *is implementable if and only if the following four conditions are satisfied*

[18] Note that the interim prospect is the expectation of a linear transformation of the SCF, hence it is not a well defined probability. In particular, its domain lies on [-1,1].

1. $P_1(1,1) = P_1(\theta,\theta)$
2. $P_1(1,\theta) \geq P_1(\theta,1)$
3. $P_1(1,1) \geq (P_1(\theta,1) + P_1(1,\theta)) / 2$
4. $P_1(1,1) \leq (P_1(\theta,1)\,\theta + P_1(1,\theta)) / (1+\theta)$.

Proposition 2's proof is an immediate consequence of imposing the conditions for truthtelling. The first condition follows from requiring that a high type does not have an incentive to deviate by declaring he is a low type together with a low type not having incentives to deviate by declaring he is a high type. The rest of the conditions follow from considering the remaining deviations. Henceforth P denotes the set of implementable mechanisms.

Observe that the mechanism treats an enthusiastic and an apathetic voter in exactly the same way ($P_1(1,1) = P_1(\theta,\theta)$).[19] This highlights the fact that the first best allocation (the one that maximizes the sum of ex-ante utilities) can never be achieved since it requires interpersonal comparisons of utility. That is, it requires favouring those voters with stronger preferences and this can never be incentive compatible. The last three conditions imply that the interim utilities should be convex. In particular, they require the interim prospect on an issue to be weakly increasing in the declaration on that issue: $P_1(1,1) \geq P_1(\theta,1)$ and $P_1(1,\theta) \geq P_1(\theta,\theta)$.

[19] The symmetry across issues property plays a relevant role for this result to hold true. The next example shows that dropping such property may be critical in the case with discrete preferences:

There is only one voter (i=1), and there only two issues (n=1,2). The player's valuation θ_1^1 and θ_2^1 are stochastically independent and uniformly distributed on {1,2}. The following SCF is strategy-proof but is not homogeneous of degree zero (i.e. it allocates a different outcome to the players (1,1) and (2,2)):

$p_1(1,1)=1$	$p_2(1,1)=0$	$p_1(0,1)=0$	$p_2(0,1)=1$
$p_1(1,0)=1$	$p_2(1,0)=0$	$p_1(2,2)=0$	$p_2(2,2)=1$

I am indebted to Tilman Borgers for bringing this fact to my attention.

In Hortala-Vallve (2009) we generally characterise all implementable mechanisms in multidimensional settings with no transfers and show that the «equal treatment of proportional voters» holds in general whenever we have a continuous support.

5.2. *Is qualitative voting optimal?*

From the viewpoint of the designer of the mechanism it is reasonable to ask if the voting rule he would like to implement is the best one under the «veil of ignorance». That is, if we weight all the possible combinations of types according to their prior distribution, does a given voting rule achieve the best possible allocation?

As Holmstrom and Myerson (1983) first pointed out, «the proper object for welfare analysis in an economy with incomplete information is the decision rule, rather than the actual decision or allocation ultimately chosen [...] a decision rule is efficient if and only if no other feasible decision rule can be found that may make some individuals better off without ever making any other individuals worse off.» In our setting this means that we do not have to compare the set of final allocations but the set of implementable mappings from preference profiles to allocations (i.e. implementable mechanisms). It would be useless to provide a welfare analysis that ignored incentive compatibility constraints because strategic manipulation of privately held information will almost surely lead to a different allocation than the expected one.

The welfare criteria we are interested in is the set of implementable mechanisms that reach a Pareto optimal allocation at the ex-ante stage.[20] Voter i's ex-ante utility from $p \in P$ is denoted $u_i(p) := E_\theta \{u(p(\theta^i, \theta^{-i})\, \theta^i)\}$.

Definition 3 *An* ex-ante efficient *mechanism* $p : \Theta^I \to X$ *is an implementable mechanism such that there does not exist any other implementable mechanism that makes some voters better off and no voters worse off, i.e.*

$$p \text{ is ex-ante efficient} \quad \Leftrightarrow \quad \nexists\ \hat{p} \in P \text{ such that } u_i(\hat{p}) \geq u_i(p) \text{ for all } i=1,\dots,I \text{ and } u_i(\hat{p}) > u_i(p) \text{ for some } i \in \{1,\dots,I\}.$$

Definition 4 *A mechanism is said to be* optimal *if its associated direct revelation mechanism is reasonable and ex-ante efficient.*

[20] Our definition of ex-ante efficiency corresponds to the notion of ex-ante incentive efficient in Holmstrom and Myerson (1983).

It is important for a mechanism to be ex-ante efficient since then it will be stable in the sense that voters will never want to jointly deviate and choose a different decision rule. This argument also holds at the interim stage: we want mechanisms to be robust once agents (privately) know their types. Ex-ante efficiency implies interim efficiency, hence our welfare criteria will also imply the stability of the voting rule at the interim stage.

The night out example described above illustrates that MR is in some cases not interim efficient. In that example, John and Anna had incentives to concede on their least preferred issue and both go to the Italian restaurant and the comedy film. It follows that MR is not ex-ante efficient and that both friends may unanimously agree on resolving their dissenting issues through alternative methods.

The assumption that the intensity of the preferences towards each issue can only take two values (θ and 1) becomes crucial at this point. It allows us to write the interim prospects in terms of a finite number of parameters and, given that we restricted the analysis to reasonable mechanisms, the number of parameters is manageable. The optimal mechanisms are simply those that maximise the ex-ante utility of any single voter subject to the four constraints in Proposition 2. The detailed analysis of the resulting linear program is left to the appendix.

Theorem 1 In a setting with two issues and two voters, QV is optimal. Moreover, MR is not optimal.

QV replicates the only ex-ante efficient and reasonable mechanism; no other mechanism can do better.

In the three voters case we have seen that QV has two equilibria: one that replicates the MR outcome and one that allows strong minorities to decide over weak majorities. The next Theorem tells us when is the second equilibrium ex-ante efficient.

Theorem 2 In a setting with two issues and three voters, whenever the values of the various issues are «different enough» (i.e. $\theta \in (0, 1/3)$), QV is ex-ante optimal. Moreover, in that case MR is not optimal.

What do we mean by issues being «*different enough*»? Recall that when we described the simplified model we denoted the relative valuation of a low issue with respect to a high one as θ. QV is optimal whenever the valuation of the high issue is at least three times the one of the low issue, $\theta \in (0, 1/3)$. In other words, it is optimal to implement the will of an enthusiastic minority as long as the majority does not oppose the preference of the minority too strongly —agents want to commit to use such a rule before knowing their preferences so that their possibly strong views are not silenced by indifferent majorities.

In the interval $\theta \in ((1/3),(1/2))$ the allocation achieved by the third equilibrium replicates the optimal allocation. For $\theta \in ((1/2),1)$ MR achieves the optimal allocation. Proofs are provided in the appendix. Note that the costs of the incentive compatibility are captured precisely in the interval $\theta \in ((1/3),(1/2))$: from an ex-ante perspective (and regardless of incentive constraints) it is optimal for a strong minority to decide over a weak majority when θ is below one half. In the same way that in the Myerson-Satterthwaite Theorem (1983) we observe that efficient trade in a bilateral asymmetric information setting may not always occur due to incentive compatibility constraints, in our setting we have that, in order to achieve a truthful revelation of preferences, the planner needs to make sacrifices and not implement the efficient allocation when a minority feels only mildly stronger than a majority $\theta \in ((1/3),(1/2))$.

Implicit in the proposal of a voting rule that elicits the voters' intensity of preferences is the fact that gains can only arise when some voters value the issues differently. Theorem 2 reinforces this idea and shows precisely that QV is optimal only when the valuation towards the issues is *different enough*.

6. Two examples

6.1. The two voters' case: conflict resolution

A more realistic version of the night out example may take the shape of a conflict resolution situation. In this case, two parties that have agreed on all concurring issues need to resolve some dissenting ones. In this context it seems sensible not to expect the amicable behaviour we observed in the example above. Now, parties may see any concession as a loss and (given the sequential nature of bargaining) may never truthfully declare their preferred alternatives leading to the deferring of any decision.[21]

[21] The social psychology literature has largely focussed on the problem of people not declaring what they perceive as less important because there exists the risk that they will lose that issue without any compensation. See for instance Rubin et al (1986).

Imagine a family enterprise that, after being badly managed for two generations, is in a very delicate situation and decides to hire a manager or CEO to redirect their business. The new CEO's team carries out a comprehensive analysis of the situation and concludes that the image of the firm needs to be updated and two proposals are made. On the one hand a restyling of the logo will change the consumer's perception of their brand at a very low cost. On the other hand, a structural improvement of their main product line would also be beneficial to consumers' perceptions and, furthermore, it will gain the attention of the press.

The owners are against any change in their product because this is, from their point of view, the essence of their business. Similarly, they cannot contemplate a restyling of their logo because it was designed by one of their ancestors and they feel emotionally attached to it.

The negotiations between both parties are at a deadlock and, as was highlighted before, any concession is seen as a loss. Furthermore, the parties rank the issues differently. The CEO realises that the first policy is interesting given its low costs but it will have no persistent effect on the public and he sees the latter as the essential move to refloat the firm. Instead, the family owners realise that something has to change but would not like to be unfaithful to their ancestor so, above all, want to keep their logo. This is a Prisoner's Dilemma situation: whatever the opponent does any party is always better off by not conceding and declaring both issues to be equally important (it is dominant to do so). And, as it is always the case, the unique equilibrium is a Pareto dominated one.

QV allows the voters to unlock the negotiation and non-cooperatively choose the Pareto optimal allocation. Let us analyse its logic: the CEO and the family are endowed with V votes each and invest all votes in their preferred issue. The reason being that, given the binary nature of the situation, winning one issue implies losing the remaining one. Hence, the optimal strategy is to make sure that the most preferred issue is not lost.

Note that a particular feature of the conflict resolution situation (voters' preferences are opposed) is that is dominant for a non-indifferent voter to invest all his votes in his preferred issue. In other words, the equilibria described in Section 3 and Theorem 1 is strategy-proof whenever the voters have opposing preferences.

6.2. The three voters' case: a committee meeting

Imagine now a religious association which is composed of three equally sized factions. Each faction delegates its rights to one of its members and these representatives meet in an annual committee to update the association's position in two major biological scientific advances: human cloning and the use of stem cells. Imagine that each of the representatives has no clue about his opponents' preferences but privately know his own. The most progressive representative has no strong position on either of the issues but he is in favour of both. Each of the other two strongly opposes one of the two issues and recognises that the positive aspects of the other one outweighs his moral prejudices and hence favours it. The next diagram summarises their positions:

	Human cloning	Use of stem cells
R1	*agree*	*agree*
R2	*strongly disagree*	*agree*
R3	*agree*	*strongly agree*

If they vote using MR, both issues are approved: a weak majority imposes its will over a strong minority. Is that situation optimal? We have just shown that from an ex-ante perspective (i.e. before voters know what they are going to vote) the MR outcome may not be optimal. If the difference between the strength of the strongly disagree and the agree positions is wide enough, it is optimal to allow the enthusiastic minorities to decide over the apathetic majorities.

Following the analysis above, the first representative splits his votes evenly, the second invests all of them in the first issue and the third does the same in the second one (as depicted in the table below).

	Human cloning	Use of stem cells
R1	$V/2$	$V/2$
R2	$-V$	0
R3	0	$-V$

The outcome is now the opposite to the one before, both issues are dismissed and the overall welfare is strictly higher than the one obtained through MR whenever the minority views are intense enough.

7. Discussion

The equilibrium of the voting game is not driven by the non-divisibility of points or the binary nature of preferences. Whenever we consider preferences to belong to the interval [-1,1] with independent and uniform priors, voters still follow the strategy described above: they invest all their votes in their most preferred issue or, in the case with three voters, they may split their votes evenly.[22]

Conversely, the optimality analysis rested heavily on the binary nature of the preferences and the uniform and pairwise independent priors. It seems natural to relax the latter assumptions and check whether the main optimality results are affected by such a change. A more precise knowledge of the opponents' preferences may lead to the non-existence of pure strategy equilibria in the game induced by QV. The intuition is the following: for the voting profiles to be an equilibrium in a complete information framework, no voter should invest a single vote in an issue he is going to lose; consequently, a single vote should be sufficient to win any issue and overcoming the single vote invested by an opponent will occur almost surely.[23] Hence, relaxing the priors may lead to some critical problems in the applicability of QV and in its optimality properties.[24]

Briefly, we have seen that more skewed priors may lead to voters becoming more strategic. Consequently, it is more difficult to achieve truthful revelation of preferences and these interactions may outweigh the welfare gains we expect from the use of QV. This contrasts with the behaviour we observe under MR where voters always declare their type truthfully. In other words, MR is robust to any possible specification on the preferences' prior distributions.

It is largely the above observation that leads us to the general analysis of the intensity problem in a related paper where we characterise all the implementable mechanisms that are robust to any specification of the priors (when no transfers are allowed) as those that induce indirect utilities that are homogeneous of degree one and convex. We then proceed to show that the utilitarian allocation (the one that maximises the sum of utilities) is not achievable. When we impose the unanimity property —an issue must

[22] Formal proofs of these statements are available from the author upon request.

[23] In general it is also true that the situation where ties occur in all issues is not an equilibrium.

[24] This may contrast with the intuition derived from Cremer and McLean (1988) that correlation allows the attainment of an efficient allocation. The result does not follow in our setting because correlation enhances the strategic interaction between individuals without introducing arbitrarily large penalties associated with lying (recall that we are not allowing transfers). Jackson and Sonnenschein (2007) provide an example that illustrates how the correlation on the intensity between the issues affects the gains we expect from linking decisions: perfect positive correlation collapses the problem into a one-dimensional one; conversely, perfect negative correlation is the best possible scenario for QV.

be approved (dismissed) with certainty if all voters support (oppose) it— we show that there exists no robust mechanism that is sensitive to the intensity of voters' preferences. These negative results may explain the absence of such mechanisms in the real world. Still, they do not undermine the value of characterising mechanisms and situations where strict Pareto improvements are possible.

The usual way for political parties to express the intensity of their preferences is through logrolling. Logrolling is defined as the exchanging of votes among legislators to achieve the approval or dismissal of the issues that are of interest to one another. Heuristically we could say that QV is related to logrolling in the same way monetary economies are related to barter. It eases the ways through which agents can express their willingness to influence given that it does not require a double coincidence of wants. Furthermore, it seems reasonable to expect that this increased freedom in the available strategies should prevent agents trading their votes because under QV a vote for an issue can never be worthless since it can be unilaterally moved to a more relevant issue.

The problem of modelling such phenomena theoretically arises because they usually occur in a situation where agents have some knowledge of their opponents' preferences but there is still scope for the understatement of one's preferences and, of course, the violation of the agreement once it is made.[25] The latter can be easily overcome via reputational arguments but the former generates major difficulties and remains an area of interest for future research.

Finally, the selection of the agenda is shown to be a central issue when analysing QV and is one of the most important problems that arises in any negotiation. The introduction of a new bill can drastically change the action taken by a particular individual. The question is then, how, by who and when should the issues be selected? There is a clear incentive to manipulate the agenda in order to induce particular outcomes and bundle issues that benefit particular groups.[26] Nevertheless, the literature lacks trac-

[25] In Hortala-Vallve (2009) I analyse in detail vote trading agreements.

[26] As an example of the scope of such a problem see Metcalfe (2000). In the context of criminalising bribery at an international level between OECD countries, he shows how the agreement on the agenda monopolised the negotiations for twelve years. He also emphasizes the perverse effect that the introduction of a divisive issue has in a negotiation: it creates a conflict between two factions that strongly disagree on the outcome of such issue and prevents any agreement being reached on the remaining ones. In a different setting Dutta y cols. (2004) define and prove the existence of an equilibrium for agenda formation when one alternative has to be selected out of many —other studies on agenda setting can be found therein.

table models of agenda setting and it remains unclear the amount of information on opponents' preferences that would be required for successful manipulation. We have therefore restricted ourselves to cases in which the agenda is exogenously given.

8. Conclusion

We have proposed an alternative to the usual voting rule which is simple and allows voters to express their willingness to influence. The proposed mechanism seems to be the most natural extension to MR and we have proved it to be not only superior to MR but also to be a mechanism that achieves the best possible allocation and induces truthful revelation of the voters' preferences in some settings. Motivated by our initial quotation we have extended the use of purely economic concepts into the political system. Nevertheless, the need for further analysis of this and similar mechanisms is clear.

The main findings of this article are summarised in its two theorems: (1) QV unlocks conflict resolution situations allowing each of the opponents to trade off their voting power between the various divergent issues; and (2) in a situation with more than two voters, QV allows very enthusiastic minorities to decide on those issues that the majorities are mostly indifferent towards.

Given the appealing properties of QV as a simple alternative to MR that could easily be used in the real world we are currently analysing its performance in an experimental setting.

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Appendix

Proof of Part 1 of Lemma 1

Given uniform and independent priors we can restrict our attention without loss of generality to voters with positive preferences.

Assume that there is an equilibrium where non-indifferent voters use different strategies. That is, where a voter that prefers the first issue invests γ on his preferred issue and a voter that prefers the second issue invests w on his preferred issue ($\gamma \neq w$). Finally, an indifferent voter invests γ_{ind} on the first issue (without loss of generality we assume that $\gamma_{ind} \geq V/2$). Once again, the assumptions on priors imply that γ, $w \geq V/2$ and $\gamma \geq \gamma_{ind}$, $w \geq V - \gamma_{ind}$. We now show that the proposed strategies cannot in fact form an equilibrium because an indifferent voter always has incentives to deviate.

Any voter can face thirty six possible situations on each issue depending on the strategies played by his opponents. In some situations the votes cast by his opponents are higher or equal than zero in which case, regardless of his strategy, the issue is approved. Similarly, if the invested votes are smaller or equal than $-V$ the issue is dismissed. The table below depicts these situations with a positive and negative sign, respectively. The remaining cells capture the total number of votes cast by voters two and three:

ISSUE 1

γ	+	+	+	+	+	+
γ_{ind}	$\gamma_{ind} - \gamma$	+	+	+	+	+
$V - w$	$V - w - \gamma$	$V - w - \gamma_{nd}$	+	+	+	+
$-(V - w)$	$-V - w - \gamma$	$-V - w - \gamma_{ind}$	$-2(V - w)$	+	+	+
$-\gamma_{\text{ind}}$	−	−	$-V - w - \gamma_{ind}$	$V - w - \gamma_{ind}$	+	+
$-\gamma$	−	−	$-V - w - \gamma$	$V - w - \gamma$	$\gamma_{ind} - \gamma$	+
	$-\gamma$	$-\gamma_{ind}$	$-(V - w)$	$V - w$	γ_{ind}	γ

We can replicate the same table for Issue 2 by considering the following set of actions $\{-w, -(V - \gamma_{ind}), -(V - \gamma), V - \gamma, V - \gamma_{ind}, w\}$. We can then compute the final allocation in each possible situation whenever voter one follows the three possible strategies. That is, whenever he invests $(\gamma_{ind}, V - \gamma_{ind}$, $(\gamma, V - \gamma)$ or $(V - w, w)$. In order to compute the expected interim payoffs we define the following parameters:

a = 1	$\Leftrightarrow$	$V - \gamma - w + \gamma_{ind} \geq 0$	a = –1	$\Leftrightarrow$	$V - \gamma - w + \gamma_{ind} < 0$
b = 1	$\Leftrightarrow$	$-2V + 2w + \gamma_{ind} > 0$	b = –1	$\Leftrightarrow$	$-2V + 2w + \gamma_{ind} \leq 0$
c = 1	$\Leftrightarrow$	$2V - \mathrm{w} - 2\gamma_{ind} \geq 0$	c = –1	$\Leftrightarrow$	$2V - w - 2\,\gamma_{ind} < 0$
d = 1	$\Leftrightarrow$	$2V - \gamma - \mathrm{w} - \gamma_{ind} \geq 0$	d = –1	$\Leftrightarrow$	$2V - \gamma - w - \gamma_{ind} < 0$
e = 1	$\Leftrightarrow$	$-V + 2\gamma - \gamma_{ind} > 0$	e = –1	$\Leftrightarrow$	$-V + 2\gamma - \gamma_{ind} \leq 0$
B = 1	$\Leftrightarrow$	$-2V + \gamma + 2w > 0$	B = –1	$\Leftrightarrow$	$-2V + \gamma + 2w \leq 0$
C = 1	$\Leftrightarrow$	$2V - 2\gamma - w \geq 0$	C = –1	$\Leftrightarrow$	$2V - 2\gamma - w < 0$

Weighting each possible situation by its probability[27] we have that the expected payoffs (we denote them $\Pi(\cdot)$) from playing the three possible strategies are

$$\Pi(\gamma_{ind}, V - \gamma_{ind}) \cdot 64 = 58 + 2a + b + 4c + 2d + e$$
$$\Pi(\gamma, V - \gamma) \cdot 64 = 60 - 4a + 4d - 3e + B + 2C$$
$$\Pi(w, V - w) \cdot 64 = 63 + 4a - 4b - 4c - 4d - 2B - C$$

Now we just need to consider all possible combinations of parameters to check whether it is strictly better to deviate. Analysing the previous inequalities we know the following set of inequalities needs to be satisfied: [28]

$a \geq C \geq d \geq c$ and $B \geq b$.

Whenever a=-1 an indifferent voter is strictly better off by playing $(\gamma, V - \gamma)$. Hence, for the proposed strategies to be an equilibrium a should be equal to one.

Repeating the previous reasoning for d = -1 we can also see that an indifferent voter has incentives to deviate by playing $(V - w, w)$. Thus, C = d = 1.

Now assume that c = -1. In that case, the expected interim payoffs are equal to $\Pi(\gamma_{ind}, V - \gamma_{ind}) \cdot 64 = 58 + b + e$ and $\Pi(w, V - w) \cdot 64 = 66 - 4b - 2B$. Note that it is not strictly better to deviate only when b = e = B = 1. It can be easily shown that d = 1 and b = 1 imply that $w > \gamma$, but d = 1 and e = 1 imply that $V + \gamma - w - 2\,\gamma_{nd} > 0$. The latter inequality cannot hold when $w > \gamma$. Hence, in equilibrium, a = C = d = c = 1.

[27] Given the uniform and independent priors, all columns (alternatively rows) occur with probability (1/8) except columns two and five which occur with probability (1/4).

[28] For instance, $a = -1 \Rightarrow C = d = c = -1$.

Suppose now that B = 1. First note that in that situation e should be equal to –1 because e = 1 implies (together with d = 1) that $w > \gamma$ and this is not compatible with $V + \gamma - w - 2\gamma_{ind} > 0$ (this inequality results from combining C = 1 and B = 1). Thus e = -1. Nevertheless, in that situation a non-indifferent voter that prefers issue 2 is better off by deviating and playing $(\gamma_{ind}, V - \gamma_{ind})$. Hence, in equilibrium B = b = –1.

e = 1 implies, as before, that a non-indifferent voter that prefers issue 2 is better off by deviating and playing $(\gamma_{ind}, V - \gamma_{ind})$. And finally, e = –1 achieves an allocation which is identical to having all voters splitting evenly their voting power (hence it is essentially a situation where $\gamma = w = \gamma_{ind}$ and all values are close enough to $(V/2)$, i.e. $2V - 3\gamma \geq 0$).

Finally, the independent and uniform priors imply that the number of votes invested in the most preferred issues should be at least as big as the number of votes invested in the least preferred one, i.e. $\gamma \geq V/2$.

Proof of Part *2* of Lemma *1*

This proof is analogous to the previous one. Assume that there is an equilibrium (γ, γ_{ind}) such that indifferent voters do not split their voting power evenly. That is, an equilibrium that reaches a different allocation to $(\gamma, V/2)$. Without loss of generality we assume that $\gamma_{ind} > V/2$. Given that the only equilibrium with $\gamma = \gamma_{ind}$ is $(V/2, V/2)$ we have that $\gamma > \gamma_{ind} > V/2$. As before, the uniform and independent priors allow us to do our analysis from the perspective of voter one and we assume that he has positive preferences (i.e. he desires the approval of both issues).

We should now replicate the table above with the thirty six possible situations that a voter can face on each issue depending on the strategy played by both his opponents.[29]

We can now compute the final allocation in each possible situation whenever voter one follows the proposed strategy and whenever he unilaterally deviates and invests $(V - \gamma_{ind})$ votes in the first issue. As noted in the main text, we want to consider a deviation where voter one, realising that both his opponents invest more voting power on the first issue, deviates and casts more votes on the second one. Furthermore, the conside-

[29] For issue 1 the set of actions is $\{-\gamma, -\gamma_{ind}, -(V - \gamma), (V - \gamma), \gamma_{ind}, \gamma\}$ and for issue 2 the set of actions is $\{-\gamma, -(V - \gamma_{ind}), -(V - \gamma), (V - \gamma), (V - \gamma_{ind}), \gamma\}$.

red deviation does not change his payoff when he faces non-indifferent voters. In order to compute the expected interim payoffs we define the following parameters:

a = 1	⇔	$V - 2\gamma + \gamma_{ind} \geq 0$	a = –1	⇔	$V - 2\gamma + \gamma_{ind} < 0$
b = 1	⇔	$-2V + 2\gamma + \gamma_{ind} > 0$	b = –1	⇔	$-2V + 2\gamma + \gamma_{ind} \leq 0$
c = 1	⇔	$2V - \gamma - 2\,\gamma_{ind} \geq 0$	c = –1	⇔	$2V - \gamma - 2\gamma_{ind} < 0$
d = 1	⇔	$-2V + 3\,\gamma_{ind} \leq 0$	d = –1	⇔	$-2V + 3\gamma_{ind} > 0$

Weighting each possible situation by its probability we have that the expected payoffs when non-deviating and deviating are respectively [(27 – 2b + 4c – a) + (31 + 2a + b)] / 64 and Π + [8 – 4c + 4d] / 64. Now we just need to consider all possible combinations of parameters to check whether it is strictly better to deviate.

Whenever d = 1 or c = -1 it is strictly better to deviate. Instead, when d = –1 and c = 1 both strategies yield the same expected payoff. Nevertheless in that situation some of the hypotheses are violated: for b = –a = 1 and b = –a = –1, (γ, γ_{ind}) is essentially equal to $(\gamma, V/2)$; instead, when a = b = 1, (γ, γ_{ind}) does not constitute an equilibrium because a non-indifferent voter that prefers issue one is strictly better off playing γ_{ind} votes on the first issue; finally, the case a = b = -1 can never happen.

Proof of Proposition 1

This proof is analogous to the previous ones. Given that indifferent voters invest V/2 votes in each issue we have that all possible combinations votes cast on any of the issues by two voters that follow the strategy $(\gamma,(V/2))$ are depicted in the matrix below:

γ	+	+	+	+	+	+
$V/2$	$V/2-\gamma$	+	+	+	+	+
$V-\gamma$	$V-2\gamma$	$V/2-\gamma$	+	+	+	+
$-(V-\gamma)$	–	$-3V/2+\gamma$	$-2(V-\gamma)$	+	+	+
$-V/2$	–	–	$-3V/2+\gamma$	$V/2-\gamma$	+	+
$-\gamma$	–	–	–	$V-2\gamma$	$V/2-\gamma$	+
	$-\gamma$	$-V/2$	$-(V-\gamma)$	$V-\gamma$	$V/2$	γ

As we did earlier, we define the following four parameters:

a = 1	⇔	$v \geq 2\gamma - V$	a = –1	⇔	$v < 2\gamma - V$
b = 1	⇔	$v \geq \gamma - V/2$	b = –1	⇔	$v < \gamma - V/2$
c = 1	⇔	$v > (3/2)\, V - \gamma$	c = –1	⇔	$v \leq (3/2)\, V - \gamma$
d = 1	⇔	$v > 2V - 2\gamma$	d = –1	⇔	$v \leq 2V - 2\gamma$

where v indicates the number of votes invested in issue one by the remaining voter. Without loss of generality we assume that this voter has positive preferences and strictly prefers the first issue.

$(\gamma, V/2)$ is an equilibrium if and only if it is optimal for the remaining voter to invest exactly γ votes on the first issue (i.e. v = γ should be optimal).

We proceed by defining all possible cases such that the conditions that define the four parameters are well ordered. For instance, whenever $\gamma > 5V/6$ we have that $0 \leq 2V - 2\gamma \leq \gamma - V/2 \leq 3V/2 - \gamma \leq 2\gamma - V \leq V$ and it can easily be shown that $v = \gamma$ is an optimal response for voter one. Hence, $(\gamma, V/2)$ is a symmetric equilibrium as long as $\gamma \in (5V/6, V]$. This set of equilibria are essentially identical to $(V, V/2)$.

A further analysis shows that there exists no symmetric equilibrium where $\gamma \in (3V/4, 5V/6]$. The case in which $\gamma = 3V/4$ implies that $0 < \gamma - V/2 < 2V - 2\gamma = 2\gamma - V < 3V/2 - \gamma < V$ and a symmetric equilibrium can be sustained if and only if $\gamma = 1/2$. If $\gamma < 1/2$, voter one prefers investing more voting power on his preferred issue, whereas he prefers to split his votes more equally whenever $\gamma > 1/2$. Hence we conclude that (3V/4,V/2) is an equilibrium if and only if $\gamma = 1/2$. Moreover, that equilibrium can be sustained by any $\gamma \in (2V/3, 3V/4]$.

Finally, $\gamma \in (V/2, 2V/3)$ can constitute a symmetric equilibrium only when $\gamma \geq 1/2$; when $\gamma < 1/2$, a non-indifferent voter knows that by deviating and investing all of his voting power on his preferred issue he gains that issue when he is confronted with an indifferent voter and a low one (but he loses it if he invests γ votes). This equilibrium reaches the same allocation as MR. In fact, (V/2,V/2) is trivially an equilibrium for any γ because any voter is equally pivotal with any number of votes (in particular with γ = V/2).

Proof of Theorem 1

Any direct mechanism is defined by 512 parameters. That is, all possible combinations of both voters' types multiplied by the number of issues we are considering. Restricting the analysis to reasonable mechanisms renders the problem tractable and simplifies the analysis into six parameters; we need to define the mechanism only on a particular issue when both voters' preferences on that issue are opposed and this can be done regardless of the sign of the remaining issue.

More precisely, the neutrality property defines the value of the mechanism whenever voters have analogous preferences (i.e. whenever both voters coincide on how strongly they prefer each issue) and allows us to focus on positively valued issues (when the agent we analyse wants the approval of the issue). The symmetry across issues allows us to focus on a particular issue (say, issue one) and the neutrality across issues property reduces the possible types we have to analyse to four because the mechanism has to be invariant with respect to the sign of the remaining issue. Finally, unanimity implies that we only have to consider the cases when the opponent wants the dismissal of issue one. The next table depicts the six parameters that uniquely define any mechanism given the properties above:

$(1,\theta)$	*1/2*	*A*	*B*	*C*
$(\theta,1)$		*1/2*	*D*	*E*
$(1,1)$			*1/2*	*F*
(θ,θ)				1/2
	$(-1,\theta)$	$(-\theta,1)$	$(-1,1)$	$(-\theta,\theta)$

Note that these parameters are the probabilities of approving an issue, hence they lie in the interval [0,1]. We define the interim prospects given the four possible declarations as $P_1\,(1,\theta)$, $P_1\,(\theta,1)$, $P_1\,(1,1)$ and $P_1\,(\theta,\theta)$. For instance,

$$P_1(1,\theta) = 2\{\mathrm{E}_{\tilde{\theta}}(\mathrm{p}(1,\theta),\tilde{\theta})\} - 1 = 2\frac{1}{8}\left(\frac{1}{2} + \mathrm{A} + \mathrm{B} + \mathrm{C} + 4\right) - 1.$$

The optimal (reasonable and ex-ante efficient) mechanism is the one that maximises the ex-ante utility subject to the truthtelling constraints (Proposition 2) and the feasibility ones (the six parameters need to belong to the interval [0,1]). The program reads as follows

$$\max_{A,B,C,D,E,F \in [0,1]} u^i (p) = 8 \, [\, 3 + A + C - D + F + (\, 4 - A - B + E + F \,)\, \theta]$$

$$\text{subject to} \quad \begin{aligned} &-B + C - D + E + 2F - 1 = 0 \\ &2A + B + C - D - E - 1 \geq 0 \\ &-6B - 2C - 6D - 2E + 4F + 6 \geq 0 \\ &-A - 2B - C - D + F + 2 + (\, A - B - 2D - E + F + 1 \,)\, \theta \leq 0 \end{aligned}$$

Solving this linear program we get that A = C = B = 1, D = E = 0 and F = ½. This allocation is the same as the one achieved by QV, hence QV is optimal.[30]

Proof of Theorem 2

Any direct mechanism is now defined by 8192 parameters. Restricting the analysis to reasonable mechanisms renders the problem tractable and simplifies the analysis into 44 parameters belonging to the interval [0,1]. The following tables define the parameters depending on the preferences of each individual. Note that since we have three voters the final allocation should be a three dimensional table. Hence, in order to depict it we provide four tables each one corresponding to a different preference profile of voter one (as we assume throughout, voter one has positive preferences towards both issues).

[30] Note that IC implies that players that are indifferent between the issues should be treated analogously at the interim stage whether they hold strong or weak preferences. We have now proved that this is not only the case at the interim stage but also at the ex-post stage. In other words, the optimal implementable SCF does not undertake ex-post interpersonal comparisons of utility.

$\theta^1 = (1,\theta)$,

$(1,\theta)$	A	B	C	D	1	1	1	1
$(\theta,1)$	E	F	G	H	1	1	1	1
$(1,1)$	I	J	K	L	1	1	1	1
(θ,θ)	M	N	O	P	1	1	1	1
$(-\theta,\theta)$	1 – M	Q	R	S	P	L	H	D
$(-1,1)$	1 – I	T	U	R	O	K	G	C
$(-\theta,1)$	1 – E	V	T	O	N	J	F	B
$(-1,\theta)$	1 – A	1 – E	1 – I	1 – M	M	I	E	A
	$(-1,\theta)$	$(-\theta,1)$	$(-1,1)$	$(-\theta,\theta)$	(θ,θ)	$(1,1)$	$(\theta,1)$	$(1,\theta)$

$\theta^1 = (\theta,1)$,

$(1,\theta)$	E	F	G	H	1	1	1	1
$(\theta,1)$	1 – V	a	b	c	1	1	1	1
$(1,1)$	1 – T	d	e	f	1	1	1	1
(θ,θ)	1 – Q	g	h	i	1	1	1	1
$(-\theta,\theta)$	1 – N	1 – g	j	k	i	f	c	H
$(-1,1)$	1 – J	1 – d	l	j	h	e	b	G
$(-\theta,1)$	1 – F	1 – a	1 – d	1 – g	g	d	a	F
$(-1,\theta)$	1 – B	1 – F	1 – J	1 – N	1 – Q	1 – T	1 – V	E
	$(-1,\theta)$	$(-\theta,1)$	$(-1,1)$	$(-\theta,\theta)$	(θ,θ)	$(1,1)$	$(\theta,1)$	$(1,\theta)$

$\theta^1 = (1,1)$,

$(1,\theta)$	I	J	K	L	1	1	1	1
$(\theta,1)$	1 – T	d	e	f	1	1	1	1
$(1,1)$	1 – U	1 – l	n	o	1	1	1	1
(θ,θ)	1 – R	1 – j	p	q	1	1	1	1
$(-\theta,\theta)$	1 – O	1 – h	1 – p	r	q	o	f	L
$(-1,1)$	1 – K	1 – e	1 – n	1 – p	p	n	e	K
$(-\theta,1)$	1 – G	1 – b	1 – e	1 – h	1 – j	1 – l	d	J
$(-1,\theta)$	1 – C	1 – G	1 – K	1 – O	1 – R	1 – U	1 – T	I
	$(-1,\theta)$	$(-\theta,1)$	$(-1,1)$	$(-\theta,\theta)$	(θ,θ)	$(1,1)$	$(\theta,1)$	$(1,\theta)$

$\theta^1 = (\theta,\theta)$,	$(-1,\theta)$	$(-\theta,1)$	$(-1,1)$	$(-\theta,\theta)$	(θ,θ)	$(1,1)$	$(\theta,1)$	$(1,\theta)$
$(1,\theta)$	M	N	O	P	1	1	1	1
$(\theta,1)$	1 – Q	g	h	i	1	1	1	1
$(1,1)$	1 – R	1 – j	p	q	1	1	1	1
(θ,θ)	1 – S	1 – k	1 – r	s	1	1	1	1
$(-\theta,\theta)$	1 – P	1 – i	1 – q	1 – s	s	q	i	P
$(-1,1)$	1 – L	1 – f	1 – o	1 – q	1 – r	p	h	O
$(-\theta,1)$	1 – H	1 – c	1 – f	1 – i	1 – k	1 – j	g	N
$(-1,\theta)$	1 – D	1 – H	1 – L	1 – P	1 – S	1 – R	1 – Q	M

Similarly to the proof of Theorem 1, we just need to compute the interim prospects in terms of these parameters and maximise the ex-ante utility of any of the voters subject to the truthtelling constraints. The interim prospects are proportional to:

$$P(1,\theta) = -9 + A + 2D + 2B + 2C + 2F + 2G + 2H + 2J + 2K + 2L + 2N + 2O + 2P + 2Q + 2R + S + 2T + U + V$$

$$P(\theta,1) = 2 + 2E - B + 2G + 2H - 2J - 2N - 2Q - 2T - 2V + 2i + a + 2b + 2c + 2f + 2h + 2j + l + k + 2e$$

$$P(1,1) = 9 + 2I + 2J + 2L - 2T + 2d + 2f - 2U - 2l + n + 2o - 2R - 2j + 2q - 2O - 2h - 2G - C - b + r$$

$$P(\theta,\theta) = 12 - 2L - 2f - 2R - 2j + 2O - D - 2H - 2Q - 2S + 2h + 2N - 2k + 2g - c - o + 2p + 2M + s - 2r.$$

The optimal (reasonable and ex-ante efficient) mechanism is the one that maximizes the ex-ante expected utility subject to the truthtelling constraints and the feasibility ones (i.e. the forty parameters need to belong to the interval [0,1]).

$$\max u^i(p)$$

subject to

$$P_1(1,1) = P_1(\theta,\theta)$$
$$P_1(1,\theta) \geq P_1(\theta,1)$$
$$P_1(1,1) \geq (P_1(\theta,1) + P_1(1,\theta) / 2$$
$$P_1(1,1) \leq (P_1(\theta,1)\theta + P_1(1,\theta)) / (1+\theta).$$

The end of the proof relies on writing the program in terms of the forty parameters and then, step by step, assuming whether or not any of the constraints is binding. Once this is done we are just left with some tedious (though trivial) linear programs. And it can be proved that for different values of θ the corner solution varies. More specifically, all parameters are equal to one except those specified below:

$\theta \in (0,1/3)$: R = S = U = b = c = j = k = l = 0
$\theta \in (1/3, 1/2)$: Q = R = S = T = U = j = k = l = r = 0
$\theta \in (1/2, 1)$: Q = R = S = T = U = V = j = k = l = r = 0.

A detailed analysis of such allocations tells us that they coincide with the allocations achieved by the strategies where a non-indifferent voter invests V, $3V/4$ and $V/2$ votes on his preferred issue, respectively.

The equilibria with continuous preferences and divisible votes (2 players)

We restrict the analysis to pure strategy equilibrium. In order to simplify the analysis we assume a uniform distribution on the relative intensities rather than on the preferences themselves i.e.

θ_n^i e $\{\pm 1, \pm\theta\}$

$$\Pr\left\{|\theta_1^i| > |\theta_2^i|\right\} = \left\{|\theta_1^i| > |\theta_2^i|\right\} = \frac{1}{2}$$

$\theta \sim U[0,1]$

Pairwise independence across issues and voters.

We analyse the equilibrium from the perspective of a voter with positive preferences. The interim expected payoff of voter i when he invests $v^i \in [0,V]\ \Re$ votes on the first issue is $\tilde{P}_1(v^i)\theta_1^i + \tilde{P}_2(V - v^i)\theta_2^i$ where

$$\tilde{P}_n(v^i) = \Pr\,(v^i + v^j > 0 \mid \theta_n^j < 0) + \frac{1}{2}\Pr\,(v^i + v^j > 0 \mid \theta_n^j < 0)\ n = 1{,}2$$

Simple calculations allow us to rewrite the interim expected payoff of voter i as:[31]

$$\frac{1}{2}\theta_2^{\,1} + \left(\Pr\,(v^i + v^j > 0) + \frac{1}{2}\Pr\,(v^i + v^j = 0)\right)(\theta_1^{\,1} - \theta_2^{\,1})$$

Hence, an indifferent voter is indifferent between playing any of the strategies (as was done in the binary case, we assume that he plays the undominated strategy $v^i = V/2$) and a non-indifferent voter (say he prefers issue one) wants to maximise the expression inside the curly brackets. In the case where v^{j}(·) induces an atomless distribution on $[0,V]$ it is optimal for voter one to set v^{i} = . Otherwise, if the induced distribution on the invested votes by voter j on issue one is not atomless, v^i will always be strictly higher (if possible) than the absolute value of the lowest possible value of v^j. Thus. the only equilibrium has non-indifferent voters investing all their voting power on their preferred issue. Finally note that the proof can also be applied to the case of continuous preferences and non-divisible votes. We just need to restrict the set of strategies of voter *i*.

The equilibria with continuous preferences and divisible votes (3 players)

The setting is analogous to the one described in the proof above. We just need to add the restriction that we focus our analysis on symmetric equilibrium (i.e. the three voters play the same strategy) and we further assume that voters behave equivalently regardless of the labelling or the sign of the issue.

This proof is a bit more complicated than the one above because now we need to consider whether each of them is in favour or against the approval of each of the issues in order to assign the appropriate sign to the cast votes. Once we take this into account we have that the interim prospects read as follows (v^j, $v^k \geq 0$):

$$\widetilde{P}_1(v^i) = \frac{1}{2}\Pr\,(v^j + v^k < v^i \mid \theta_1^{\,j}, \theta_1^{\,k} < 0) + \Pr\,(v^j - v^k \leq v^i \mid \theta_1^{\,j}, -\theta_1^{\,k} < 0) - \frac{1}{2}$$

$$\widetilde{P}_2(1 - v^i) = \frac{1}{2}\Pr\,(v^j + v^k > V + v^i \mid \theta_1^{\,j}, \theta_1^{\,k} < 0) + \Pr\,(v^j - v^k \leq V - v^i \mid \theta_1^{\,j}, -\theta_1^{\,k} < 0) - \frac{1}{2}$$

[31] Conditional probabilities are omitted for notational simplicity.

Note that the tie breaking rule is now playing a role because voter i just needs to equate the sum of his opponents votes whenever only one of them desires the dismissal of the issue. Given the assumption that voters play equivalently regardless of the sign of his preferences we have that v^j and (1- v^j) have the same induced distribution (the same can be said about voter k's strategy). That implies that v^j is symmetrically distributed around $V/2$. In order to simplify the notation we define $X := v^j + v^k$ (which, accordingly, is symmetrically distributed around V i.e. $\Pr(X < k) = \Pr(X > 2V - k)$ for $k \in [0,2V]$). Using such symmetry and the fact that $(v^j + (1 - v^k))$ is distributed as X, we can write the interim expected payoff for a voter that prefers issue one as follows

$$ct + = \frac{1}{2}\left(\frac{1}{2} - \theta\right) \Pr\,(X < v^i) + \left(1 - \frac{1}{2}\theta\right) \Pr\,(X \leq V + v^i)$$

First note that whenever both opponents are splitting their voting power evenly (the case of MR), voter i is indifferent between playing any of the strategies. In particular $v^i = V/2$ is a best response. Hence, a symmetric equilibrium has all voters always splitting their voting power equally among both issues.

In the remainder of the proof we show that there exists only one more (and only one) equilibrium which corresponds to the one in which non-indifferent voters invest all their voting power on their preferred issue.[32]

Any other equilibrium will have non-indifferent voters investing more than $V/2$ votes on their preferred issue. Consequently, any voter with $\theta \in [0,1/2)$ clearly invests all his voting power on his preferred issue. Suppose now that there are some voters with $\theta \in [1/2,1]$ such that $v^i < V$. Theorem 1 tell us that the optimal strategy is a well behaved function (decreasing with respect to θ) thus we can consider a parameter $\tilde{\theta} \in [1/2,1]$ such that any voter with $\theta^+ > \tilde{\theta}$ invests strictly less votes on his preferred issue ($v^i(\theta^+) < V$) and any voter with $\theta^+ < \tilde{\theta}$ sticks to the strategy $v^i = V$.

Given that both are acting optimally we have that the next two inequalities should hold:

$$(\Pr(X < V) - \Pr(X < v^i\,(\theta^-)) \times \left(\theta - \frac{1}{2}\right) \leq (\Pr(X \leq 2V) - \Pr(X \leq V + v^i\,(\theta^-)) \times (2 - \theta^-)$$

[32] The behaviour of indifferent voters does not need to be specified because they have zero measure. Nevertheless, it can be shown that their best response to any of the equilibria is splitting their voting power evenly.

$$(\Pr(X < V) - \Pr(X < v^i(\theta^-)) \times \left(\theta^+ - \frac{1}{2}\right) \leq (\Pr(X \leq 2V) - \Pr(X \leq V + v^i(\theta^-)) \times (2 - \theta^+)$$

Given that the optimal function is decreasing we have that we should consider two possible cases: (1) the function is smooth at θ (i.e. $\lim_{\varepsilon \to 0} v^i(\tilde{\theta} + \varepsilon) = V$) and (2) there is a discontinuity (i.e. . $\lim_{\varepsilon \to 0} v^i(\tilde{\theta} + ^-) = \bar{v} < V$). Consequently, taking limits as θ^- and θ^+ tend to θ in the previous inequalities lead to two possible equalities depending on the behaviour of the optimal strategy at $\tilde{\theta}$:

$$1: \quad (\Pr(X < V) - \Pr(X < v^i(\theta^-)) \times \left(\tilde{\theta} - \frac{1}{2}\right) \leq (\Pr(X \leq 2V) - \Pr(X \leq 2V) \times (2 - \tilde{\theta})$$

$$2: \quad (\Pr(X < V) - \Pr(X < \bar{v}) \times \left(\tilde{\theta} - \frac{1}{2}\right) \leq (\Pr(X \leq 2V) - \Pr(X \leq V + \bar{v}) \times (2 - \tilde{\theta})$$

Trivially, the first equality cannot be met because there is a positive measure of types playing the non-diversification strategy thus $\Pr(X = 2V) > 0$. The second case also leads to a contradiction given the following inequalities and the fact that one of them will always be strict:

$$2\tilde{\theta} - 1 \leq 2 - \tilde{\theta}$$

$$(\Pr(X < V) - \Pr(X < \bar{v}) \leq 2\,(\Pr(X \leq 2V) - \Pr(X \leq V + \bar{v}))$$

The second inequality needs some clarification. The term in brackets on the RHS accounts for all those cases in which both opponents are investing strictly more than $(V + \bar{v})$ votes (i.e. $X \in (V + \bar{v}, 2V)$). That is, those cases in which both voters have a type belonging to the interval $[0,\tilde{\theta})$. Hence this occurs with probability ρ^2 where $\rho := \Pr(\theta \in [0,\tilde{\theta}))$. Instead, the LHS accounts for those cases in which X belongs to $[\bar{v}, V)$. A necessary condition for that event is that none of the voters should invests V votes i.e. it occurs with a probability lower than $1 - \rho$. Given that θ is uniformly distributed, we know that $\rho \geq 1/2$.

Finally, we just need to see that the second inequality is strict for $\rho > 1/2$ and the first one is strict for $\rho = 1/2$.